PCT WORLD INTELLECTUAL PROPERTY ORGANIZATION
International Bureau

INTERNATIONAL APPLICATION PUBLISHED UNDER THE PATENT COOPERATION TREATY (PCT)

(51) International Patent Classification 5 : G21K 1/00	A1	**(11) International Publication Number:** WO 90/16073 **(43) International Publication Date:** 27 December 1990 (27.12.90)

(21) International Application Number: PCT/US90/03441

(22) International Filing Date: 14 June 1990 (14.06.90)

(30) Priority data:
368,246 14 June 1989 (14.06.89) US

(71)(72) Applicant and Inventor: MILLS, Randell, L. [US/US]; P.O. Box 142, Cochranville, PA 19330 (US).

(74) Agents: REYNOLDS, Leo, R. et al.; Hamilton, Brook, Smith & Reynolds, Two Militia Drive, Lexington, MA 02173 (US).

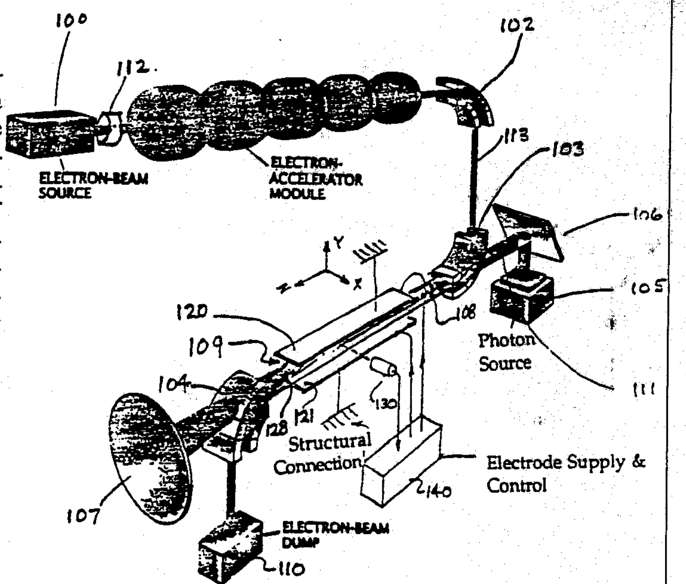
(81) Designated States: AT, AT (European patent), AU, BB, BE (European patent), BF (OAPI patent), BG, BJ (OAPI patent), BR, CA, CF (OAPI patent), CG (OAPI patent), CH, CH (European patent), CM (OAPI patent), DE*, DE (European patent)*, DK, DK (European patent), ES, ES (European patent), FI, FR (European patent), GA (OAPI patent), GB, GB (European patent), HU, IT (European patent), JP, KP, KR, LK, LU, LU (European patent), MC, MG, ML (OAPI patent), MR (OAPI patent), MW, NL, NL (European patent), NO, RO, SD, SE, SE (European patent), SN (OAPI patent), SU, TD (OAPI patent), TG (OAPI patent).

Published
With international search report.
Before the expiration of the time limit for amending the claims and to be republished in the event of the receipt of amendments.

(54) Title: APPARATUS AND METHOD FOR PROVIDING AN ANTIGRAVITATIONAL FORCE

(57) Abstract

A method and means for providing antigravitational force comprising means for providing a source of matter having a spacetime curvature opposite that of a gravitating body where opposite curvatures provide mutually repulsive antigravitational force, and method and means for applying a force on the oppositely curved matter from the source wherein in the case of the Earth, the resulting force balance provides repulsive gravitational work on the oppositely curved matter and the gravitational field of the gravitating body in the first positive curvature of spacetime. The opposite, or negative curvature of matter is provided according to a two-dimensional manifold of negative curvature derived on a novel atomic theory which is consistent with Newtonian mechanics, Maxwell's equations, atomic theory, General Relativity and the weak and strong nuclear forces and which permits the calculation of any matter, energy, or force in terms of fundamental constants of nature. According to the present invention the spacetime manifold of negative curvature is a solution of the equations and boundary conditions of the novel atomic theory comprising, in part a three-dimensional wave equation. The manifold satisfies the boundary condition that its spacetime Fourier transform contains no components which are synchronous with waves traveling at the speed of light; therefore, the manifold is non-radiative. In one embodiment, the matter of negative curvature provided by the source moves at a constant velocity at force balance to produce useful work against the gravitational field of the gravitating body to provide apparatus useful to produce levitation and propulsion. In one embodiment, electromagnetic force is converted to gravitational force and subsequently converted to a mechanical force which provides the useful propulsion or levitation. In alternative embodiments, the electromagnetic force is provided by a generator driven by steam, for example, where the energy of the strong nuclear force provides the heat energy to provide the steam. Alternatively, the heat energy is converted directly to the electromagnetic force via a generator such as a thermionic generator.



* See back of page

DESIGNATIONS OF "DE"

Until further notice, any designation of "DE" in any international application whose international filing date is prior to October 3, 1990, shall have effect in the territory of the Federal Republic of Germany with the exception of the territory of the former German Democratic Republic.

FOR THE PURPOSES OF INFORMATION ONLY

Codes used to identify States party to the PCT on the front pages of pamphlets publishing international applications under the PCT.

AT	Austria	**ES**	Spain	**MC**	Monaco
AU	Australia	**FI**	Finland	**MG**	Madagascar
BB	Barbados	**FR**	France	**ML**	Mali
BE	Belgium	**GA**	Gabon	**MR**	Mauritania
BF	Burkina Fasso	**GB**	United Kingdom	**MW**	Malawi
BG	Bulgaria	**GR**	Greece	**NL**	Netherlands
BJ	Benin	**HU**	Hungary	**NO**	Norway
BR	Brazil	**IT**	Italy	**RO**	Romania
CA	Canada	**JP**	Japan	**SD**	Sudan
CF	Central African Republic	**KP**	Democratic People's Republic of Korea	**SE**	Sweden
CG	Congo			**SN**	Senegal
CH	Switzerland	**KR**	Republic of Korea	**SU**	Soviet Union
CM	Cameroon	**LI**	Liechtenstein	**TD**	Chad
DE	Germany, Federal Republic of	**LK**	Sri Lanka	**TG**	Togo
DK	Denmark	**LU**	Luxembourg	**US**	United States of America

APPARATUS AND METHOD FOR PROVIDING AN ANTIGRAVITATIONAL FORCE

of which the following is a specification:

FIELD OF THE INVENTION

The present invention relates to methods and apparatus for providing repulsion, in particular methods and apparatus for providing antigravitational repulsive forces adapted to provide propulsion and levitation.

BACKGROUND OF THE INVENTION

The attractive gravitational force has been the subject of investigation for centuries. The development and current state of understanding is illustrated by the excerpt presented in Appendix I. Traditionally, gravitational attraction has been investigated in the field of astrophysics applying a large scale perspective of cosmological spacetime, as distinguished from currently held theories of atomic and subatomic structure. However, the disunification of the analysis of matter on the basis of scale results in inconsistencies and unpredictabilities in the resulting model, when compared to actual experimental measurements.

In Newtonian gravitation, the mutual attraction between two particles of masses m_1 and m_2 separated by a distance r is

$$F = G\frac{m_1 m_2}{r^2} \qquad (1.1)$$

where G is the gravitational constant, its value being 6.67×10^{-11} $Nm^2\ kg^{-2}$. Although Newton's theory gives a correct quantitative description of the gravitational force, the most elementary feature of gravitation is still not well defined. Which feature of gravitation is then the most important, if we were to consider the most fundamental? By comparing Newton's second law,

$$F = ma \qquad (1.2)$$

with his law of gravitation, we can describe the motion of a freely falling object by using the following equation:

$$m_i \vec{a} = m_g \frac{GM_\oplus}{r^3} \vec{r} \qquad (1.3)$$

where m_i and m_g represent respectively the object's inertial mass (inversely proportional to acceleration) and the gravitational mass (directly proportional to gravitational force), $M_\oplus$ is the gravitational mass of the Earth, and r is the position vector of the object taken form the center of the Earth. The above equation can be rewritten as

$$a = \frac{m_g}{m_i}\left(\frac{GM_\oplus}{r^2}\right) \qquad (1.4)$$

Extensive experimentation dating form Galileo's Pisa experiment to the present has shown that irrespective of the object chosen, the acceleration of an object produced by the gravitational force is the same, which from eq. (1.4) implies that the value of m_g/m_i should be the same for all objects. In other words, we have

$$m_g/m_i = \text{universal constant.} \quad (1.5)$$

In physics, the discovery of a universal constant often leads to the development of an entirely new theory. From the universal constancy of the velocity of light c, the special theory of relativity was derived; and from Planck's constant h, the quantum theory was deduced. Therefore, the universal constant m_g/m_i should be the key to the gravitational problem. The theoretical difficulty with Newtonian gravitation is to explain just why relation (1.5) exists implicitly in Newton's theory as a separate law of nature besides (1.1) and (1.2). Furthermore, discrepancies between certain astronomical observations and predictions based on Newtonian celestial mechanics exist, and they could not be reconciled until the development of Einstein's theory of General Relativity which can be transformed to Newtonian gravitation on the scale in which Newton's theory holds.

On a cosmological scale, the Theory of General Relativity is correct experimentally; however, it is based on a flawed dynamic formulation of Galileo's law. Einstein took as the basis to derive his gravitational field equations a certain kinematical consequence of that law which he called the "Principle of Equivalence" which does not provide a quantum gravitational theory. Further discussion of Einstein's "Principle of Equivalence" is provided by the excerpt of Appendix II.

Furthermore, General Relativity is a partial theory in that it deals with matter on cosmological scale, but not an atomic scale. All gravitating bodies are composed of matter and are collections of atoms which are composed of fundamental particles such as electrons, which are leptons, and quarks which make up protons and neutrons. Gravity originates form the fundamental particles. The prevailing theory of the properties of atoms and subatomic physics is quantum mechanics, and the unification of quantum theory with General Relativity is a notoriously problematic union. In contemporary theories there is no satisfactory theory of quantum gravity, and even the simplest application of quantum mechanics to the geometric structure of spacetime faces serious conceptual difficulties. Implicit to its success is that the quantum gravity theory must specify the cosmological constant. The cosmological constant can be defined most simply as the energy density of a vacuum, that is, the amount of energy in a unit volume of empty space. The gravitational equation which contains the cosmological constant is (Eq. 52.08) of Appendix VI. According to quantum mechanics the value of the cosmological constant may not be zero, and in principle, the vacuum energy density can assume any value positive or negative, and current ideas about particle physics and gravity suggest that it is quite large. The cosmological constant can be approximately measured by looking for the characteristic effects that a nonzero vacuum energy density would have on the geometric structure of spacetime as produced by Einstein's General Relativity. No such effects have been seen, and present observational limits imply that the cosmological constant is smaller that the theoretical expectations by a staggering factor of about 10^{-120}. The only way to account for this enormous discrepancy between theoretical expectation and experimental reality without rejecting either quantum mechanics, General Relativity, or both, is to assume that the parameters of nature are involved in an extraordinarily accurate and utterly mysterious conspiracy resulting in cancellation between the various contributions to the vacuum energy density. Work on the cosmological constant is based on procedures by Stephen Hawking and his collaborators for calculating a quantum mechanical wave function describing the spatial geometry of the universe. This theory states that quantum mechanical processes in our universe create and destroy other universes which are coupled to our universe and each other through wormholes. Further discussion of Hawking's theories and Sidney Coleman's extension comprising a theory of variable, α, where the wormholes affect the value of physical parameters via the α's is contained in Appendix III.

However, the Hawking approach relies on shaky formalisms and many untested assumptions. It comes up with the desired result, a zero cosmological constant, but in principle, because Coeman's scheme is a method for predicting the values of the α's and as all parameters of nature are functions of these, it is a theory of parameters. The theory has failed at providing the correct value for any other fundamental parameter, even one that is zero.

SUMMARY OF THE INVENTION

Overview of the Novel Theoretical Basis

While the inventive methods and apparatus described in detail further below may be practiced as described, the following discussion of a novel theoretical basis of the invention is provided for additional understanding.

The effects of gravity preclude the existence of inertial frames in a large region, and only local inertial frames, between which relationships are determined by gravity are possible. In short, the effects of gravity are only in the determination of the local inertial frames. The frames depend on gravity and the frames describe the spacetime background of the motion of matter; therefore, differing from other kinds of forces, gravity which influences the motion of matter by determining the properties of spacetime is itself described by the metric of spacetime. The gravitational method and apparatus of the present invention provides a metric over all scales form atomic to cosmological where the cosmological constant is zero. It is demonstrated that gravity arises form the two-dimensional manifold of the mass of fundamental particles that makes up all matter of the universe.

The novel atomic model of the present invention, which is described in detail in co-pending U. S. Patent Applications (Serial Nos. 07/341.733 and 07/345,628), both entitled ENERGY/MATTER CONVERSION METHODS AND STRUCTURES, filed on April 21, 1989 and April 28, 1989 respectively, and incorporated herein by reference, provide the correct basis to solve Einstein's field equations of General Relativity: the gravitational mass and the inertial mass of fundamental particles are equivalent. The gravitational field equations are derived form this principle in Appendices VI and VII, and atomic and cosmological gravitation are unified when the existence of the mass as a spatial two-dimensional (three-dimensional spacetime) manifold is taken into account.

The novel model of the atom is consistent with first principles, and is summarized as follows. The electron is described by the product of two angular functions which are spherical harmonics, a time harmonic function, and a radial delta function forming specific radially quantized orbitals. In order to differentiate the atomic structure of the novel atomic model from previous atomic models, these radially quantified orbitals are referred to as "Mills" orbitals. When applied to atomic structure, Mills orbitals are electron orbitals. Angular momentum and energy are conserved without violation of Maxwell's equations. The energy of an electron is the sum of its stored electric and magnetic energies. From the radial delta function, the wavelike properties of the electron arise naturally. The angular harmonics give rise naturally to spin and orbital angular momentum, spin-orbital coupling and selection rules for the absorption of emission of electromagnetic radiation. A Mills orbital is spherical, and the radius increases with the absorption of electromagnetic energy. When the electron is ionized the radius of the Mills orbital goes to infinity; the electron is a plane wave with the de Broglie wavelength.

The electric field of an electron of a Mills orbital is zero inside the orbital and is the field at a point charge at the origin outside of the orbital; thus, electron-electron repulsions are naturally eliminated in multi-electron atoms. The radii of orbitals in atoms are calculated for each orbital by setting the centripetal force equal to the sum of the coulombic and magnetic forces. Thus, the result that isolated Mills orbitals are stable where the coulombic attractive force does not cause the electron to collapse into the nucleus arises naturally. For all atoms and ions, there exists a central coulombic force acting on each orbital that is proportional to the net charge (that is the charge not cancelled by other electrons). A positive central force exists between two unpaired electrons which results in pairing in the same shell with spins opposed. Thus, the Pauli Exclusion Principle arises naturally. A diamagnetic repulsive central force exists between between paired electrons of an inner shell and an unpaired electron of an outer shell. A four-body problem does not arise because the change in the centripetal force of the inner shell electrons affected by the outer electron is exactly balanced by the Lorentzian force provided by the magnetic field of the outer shell electron. Of course, Hund's Rule arises naturally as a consequence of electrons as Mills orbitals, with angular harmonic charge density modulation, spatially orienting to form a spherically symmetric charge density function which is an energy minimum. Furthermore, as previously stated, the electric field of the electron is spherically symmetric and is that of a point charge for radial distances greater than the radius of the radial delta function and is zero inside the shell. This is the basis of chemical bonding and low temperature fusion, as described in the incorporated patent application. The nature of the chemical bond is an overlap of Mills orbitals of participating atoms to produce a minimum of the energy stored in the electric and magnetic fields. Cold nuclear fusion occurs as a result of cancellation of the coulombic repulsive force between two nuclei for shorter internuclear distances as the shell of the individual atomic electrons decreases by a quantized fraction as an energy hole is resonantly absorbed by the electron. For deuterium the resonance shrinkage energy is n/2 27.21 eV where n is an integer.

Moreover, the novel atomic model deals with electrons which comprise the negatively charged fundamental particles, leptons. However, the same principles apply to the other fundamental particles including quarks. The phenomena of wavelike properties, spin and orbital angular momentum, and selection rules are identical to those of leptons as is the boundary conditions which precludes radiation; thus, quarks have equivalent spacetime charge density functions as leptons. Also, the weak and strong nuclear forces are electromagnetic in nature as is the spin-pairing, diamagnetic, and bonding forces of leptons.

Mass exists as Mills orbitals, each of which comprises the product of two spherical harmonic angular functions, a time harmonic function, and a radial delta function. From the radial delta function which determines matter to be a two-dimensional space manifold (three-dimensional spacetime) gravitation arises. The two-dimensional manifold possesses positive curvature. This can be demonstrated by the fact that the angle sum of a geodesic triangle on the surface of a sphere exceeds 180°. Euclidean plane geometry asserts that in a plane, the angles of a triangle add up to 180°. This is the definition of a flat surface. The curvature, K, of ordinary matter with positive curvature is given by $K = 1/r_o^2$ where r_o is the radius of the radial delta function. Because essentially all ordinary matter exists in the form of Mills orbitals as leptons and quarks which exist in curved spacetime, all macroscopic configurations of ordinary matter exist in curved spacetime. Furthermore, the total curvature of spacetime is the sum of the contributions from each Mills orbital, termed a "Mills" orbital as defined in the previously filed applications.

All matter exists as Mills orbitals which comprise mass confined to a three-dimensional spacetime. The surface is spherical; thus, the spacetime is curved with constant curvature. The effect of this "local" curvature on the non-local spacetime is to cause it to be Riemannian as opposed to Euclidean. The effect is a function of the radial distance from the center of the Mills orbital for distances greater than the radius of the Mills orbital and the magnitude of the effect on non-local spacetime is proportional to the total mass of the Mills orbital. Thus, the effect on spacetime is independent of the Mills orbital radius for spacetime outside of the orbital. (According to the novel atomic theory, this feature is symmetrical with the electric field of an electron of a Mills orbital which is that of a point charge at the center for radial distances greater than the radius of the shell.) Also, the novel atomic theory embraces the postulate that the total mass of the orbital is also equivalent to the inertial mass from which the theory of General Relativity comprising Riemannian geometry of spacetime is derived.

In the theory of General Relativity, Einstein's field equations give the relationship whereby matter determines the curvature of spacetime which is the origin of gravity. The definitive form of the equations are as follows:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = - \frac{-8\pi G}{c^4} T_{\mu\nu} \qquad (1.6)$$

where

$$R_{\mu\nu} = g^{\alpha\beta} R_{\mu\alpha\nu\beta},\ R = g^{\mu\nu} R_{\mu\nu} \qquad (1.7)$$

and $T_{\mu\nu}$ is the stress-energy-momentum tensor of matter. The derivation of the above equations which are discussed in more detail in Appendices VI and VII is based on the principle of the equivalence of the inertial and gravitational mass provided by the present novel atomic model and the principle that all particles including light follow geodesics.

The Schwarzschild metric is the solution of the boundary value problem of Einstein's gravitational field equations applied to a Mills orbital, where a discontinuity in mass is equated with a discontinuity of the curvature of spacetime. Thus, the present novel atomic model and General Relativity are unified in a quantum theory of gravitation which is valid on any scale and is hereafter referred to as the Mills Quantum Gravitational Theory.

The present invention unites the three forces, electromagnetic, gravitational, and mechanical and permits their interconversion. As examples, the Meisner effect is the phenomenon whereby a superconductor of the present invention converts a gravitational force into an electromagnetic force, and the energy of the strong nuclear force is released as heat during Coulombic Annihilation (cold) Fusion as discussed in the above incorporated prior patent applications. The heat produces electricity directly via a thermionic of photovoltaic generator, or the heat produces a mechanical force via steam which turns a generator to create electricity. The electricity provides an electromagnetic, electrostatic, or magnetostatic force which by a device of the present invention warps matter into negative curvature such that an antigravitational force is produced with a gravitating body. The antigravitational force provides mechanical force, useful for propulsion or levitation.

Antigravity Methods and Means

The preferred embodiment of the present invention is a propulsion and levitation device comprising a source of matter, a means to form the matter into a spatial two-dimensional manifold of negative curvature, and a means to produce a force on the negatively curved matter where the force balances the repulsive gravitational force between the negatively curved matter and a gravitating body. In response to the force balance, the matter of negative curvature moves at constant velocity to produce useful work against the gravitational field of the gravitating body where the constant velocity including zero velocity, provides the spacetime manifold of negative curvature which is a solution to the three-dimensional wave equation whose spacetime Fourier transform does not contain components synchronous with waves traveling at the speed of light. Therefore, the manifold is nonradiative.

The spacetime manifold is based on the present novel atomic model which unites Newtonian mechanics, Maxwell's equations, atomic theory, General Relativity, and the weak and strong nuclear forces. Thus, the present invention of a propulsion and levitation apparatus comprises the conversion of an electromagnetic force to a gravitational force to a mechanical force, which provides a useful propulsion and levitation. In one embodiment the electromagnetic force is provided by a generator driven by a mechanical force provided in response to steam, where binding energy of the strong nuclear force is converted to mechanical energy indirectly via steam, for example, or directly via a thermionic generator, for example. Thus, unification and interconversion of the fundamental forces is provided by the present novel atomic model and propulsion and levitation apparatus.

In one embodiment the antigravity propulsion and levitation means comprises a means to inject particles, such as electrons, as plane waves, which serve as the matter, and further includes a guide of the plane waves. Negative curvature of the injected and guided matter is effected by applying a force on the matter. The applied force is provided by one or more of an electric field, a magnetic field, or an electromagnetic field. A second force on the negatively curved matter is applied in the direction of the gravitational force. This second force is provided by one or more of an electric field, a magnetic field or an electromagnetic field. In a preferred embodiment, the force in the gravitational direction is equal to the repulsive, antigravity force which develops between the gravitating body and the matter due to the negative curvature of the guided matter. The repulsive force of the gravitating body is then transferred to the guide (source of the second force) which further transfers the force to the attached structure to be accelerated or levitated.

In a preferred embodiment of a propulsive device, a vehicle to be accelerated comprises an antigravity levitating device and a flywheel which rotates about its axis. The antigravity force provides pure radial acceleration when the vehicle's gravitational forces are equally exceeded. An imbalance of central force applied to the vehicle will cause it to tilt. By virtue of the angular momentum of the spinning flywheel a tangential acceleration is produced which conserves angular momentum. Then high acceleration and velocity are provided by accelerating the structure along a hyperbolic path around a gravitating body such that the structure is accelerated to high velocity.

BRIEF DESCRIPTION OF THE DRAWING

These and further features of the present invention will be better understood by reading the following Detailed Description of the Invention taken together with the Drawing, wherein:

Fig. 1 is a two-dimensional graph showing the cross-section of the magnetic potential and the corresponding magnetic field lines (arrows) at a point along the channel of guiding and field generating means of Fig. 4;

Fig. 2 is a three-dimensional graph which shows the magnitude of the electric force in the z direction due to the electric potential function, xyz and the magnitude of the magnetic force in the z direction due to the magnetic potential function, xy where the electron beam propagates in the z direction;

Fig. 3 is a graph showing the spatial two-dimensional curved manifold which is the Mills orbital that propagates along the channel of the electron guide means of Fig. 4;

Fig. 4 is a drawing of a system of antigravity propulsion and levitation means according to one embodiment of the present invention;

Fig. 5 is a schematic of forces of gravitation, antigravitation, and angular momentum acting on a vehicle according to one embodiment of the present invention;

Fig. 6 is a drawing of an experimental apparatus according to one embodiment of the present invention to produce electrons of negative curvature with concomitant production of antigravity forces;

Fig. 7 is a drawing which shows the distribution of negative curvature and antigravitational forces in a relativistic electron beam following a pass through a quadrapole magnetic triplet of the apparatus of Fig. 6; and

Fig. 8 is a block diagram of an antigavitational repulsion device powered by a Coulombic Annihilation (cold) Fusion system according to one embodiment of the present invention.

DETAILED DESCRIPTION OF THE INVENTION

As previously stated, the model of fundamental particles, i.e., leptons and quarks, which provides the basis for a novel unified gravitational theory presented herein is the novel atomic model described in the previous U. S. Patent Applications entitled ENERGY/MATTER CONVERSION METHODS AND STRUCTURES which are incorporated by reference.

The novel atomic model includes electrons which comprise the negatively charged fundamental particles, leptons. However, the same principles apply to the other fundamental particles including quarks. The phenomena of wavelike properties, spin and orbital angular momentum, and selection rules are symmetrical with those of leptons as is the boundary condition which precludes radiation; thus, quarks have spacetime charge density functions equivalent to leptons. Moreover, the weak and strong nuclear forces (the spin pairing, diamagnetic, and bonding forces of leptons), are electromagnetic in nature.

DETAILED APPLICATION OF THE NOVEL THEORETICAL BASIS

Space, Elementary Particles, and the Forces

The nucleus comprises nucleons, the protons and neutrons. Each nucleon comprises three quarks as Mills orbitals analogous to the case of the electrons of the three-electron atom of the novel atomic model. The massive vector bosons W^+, W^-, and z^0 are spin-1 massive photons and are due to the weak nuclear force. W^+ and W^- carry the energy of the spin pairing weak nuclear force and z^0 carries the energy of the diamagnetic weak nuclear force.

The strong nuclear force binds quarks together in the proton and neutron, and binds the protons and neutrons together in the nucleus of an atom. The strong force is due to the field of a spin-1 particle called a gluon. The gluon is analogous to the standing traveling photonic wave that exists inside of a Mills orbital of a lepton when it absorbs a photon or an energy hole as described by the novel atomic theory. The gluon maintains the quarks in Mills orbitals and is responsible for force balance. The nucleon binding energy of the strong force which is released by Coulombic Annihilation Fusion of the novel atomic model is analogous to the energy released in the chemical bond as Mills orbitals overlap, and the total energy stored in the electric and magnetic fields is minimized.

The rules for binding of quarks by gluons is confinement which provides for conservation of angular momentum and energy, a minimization of the energy, and absence of spacetime Fourier components of the charge density functions synchronous with waves traveling at the speed of light; thus, there is no radiation. It always binds particles together into combinations that have no color. The "colors" of gluons and quarks are red, green and blue. A red quark is joined to a green and blue quark by a "string" of gluons (red + green + blue = white). Such a triplet constitutes a proton or a neutron.

Space has an intrinsic impedance of $120\pi = 377$ ohms given by the square root of the quotient of the permeability and permittivity of free space. It provides a limiting speed of c for the propagation of any wave including gravitational and electromagnetic waves. It further provides fields which match boundary conditions. Matter/energy acts on space and space acts on matter/energy. Thus, a spatial two-dimensional manifold of matter results in a gravitational field in space; a three-dimensional spacetime manifold of current gives rise to a magnetic field in space; a spatial two-dimensional manifold of charge gives rise to an electric field in space. Thus, General Relativity and Maxwell's equations are valid on any scale. Furthermore, the existence of matter with a determined mass as a three-dimensional spacetime manifold that is charged maximizes the volume of space to surface of matter ratio. This gives an energy minimum of the resulting gravitational, electric, and magnetic fields.

Matter/energy are interconnectable and are in essence the same entity with different boundary values imposed by spacetime where the matter/energy has a reaction effect on spacetime. The intricacy of the action/reaction is evident in that all matter/energy obeys the three-dimensional wave equation, and the magnetic, coulombic, photonic, and gravitational fields can be derived as a boundary value problem of the wave equation of spacetime where space provides the respective force fields for the matter/energy. That spacetime is four-dimensional is evident because the fundamental forces of gravity and coulombic attraction which are time dependent have a one over distance squared relationship which is equivalent to the distance dependence of the area of a spherically symmetric wavefront which carries the forces at the limiting speed of light provided by spacetime.

The action/reaction relationships of the third fundamental force, the mechanical force is given by Newton's Laws which provide the motion of matter including charged matter, which can give rise to gravitational, magnetic, and photonic fields. The action/reaction provided by forces in one inertial frame is given in a different inertial frame by the Lorentzian Transformations of Special Relativity which are valid for Euclidean spacetime and are a consequence of the limiting speed of light. For example, the magnetic field in on inertial frame is given as a coulombic field, in another as a consequence of their relative motion. The presence of matter causes the geometry of spacetime to deviate from Euclidean which is manifest as a gravitational field. The gravitational equation is derived for all scales from the present novel atomic model where spacetime is Riemannian.

The provision of the equivalence of inertial and gravitational mass by the present novel model of the atom permits the correct derivation of the General Theory given in Appendices VI and VII. And, the former provision of the two-dimensional nature of matter permits the unification of atomic and cosmological gravitation. The unified theory of gravitation is derived by first establishing a metric. The mathematics of the development of the metric as general tensor analysis is given in Appendices I and IV and the principles of covariance of equations are given in Appendix IV. Related remarks on General Relativity Principles are included in Appendix V.

A space in which the curvature tensor has the following form:

$$R_{\mu\nu,\alpha\beta} = K\ (g_{\nu\alpha}g_{\mu\beta} - g_{\mu\alpha}g_{\nu\beta}) \qquad (1.8)$$

is called a space of constant curvature, it is a four-dimensional generalization of Lobachevsky space. The constant K is called the constant of curvature. The curvature of spacetime will be shown to result form a discontinuity of matter confined to two spatial dimensions. This is the property of all matter as Mills orbitals. Consider an isolated Mills orbital and radial distances, r from its center. For r less than r_o where r_o is the radius of a Mills orbital, there is no mass; thus, spacetime is flat or Euclidean. The curvature tensor applies to all space of the inertial frame considered; thus, for r less than r_o, K = 0. At $r = r_o$ there exists a discontinuity of mass of the Mills orbital. This results in a discontinuity of the curvature tensor for radial distances greater than or equal to r_o. It will be shown below that the discontinuity gives rise to a boundary value problem of Einstein's gravitational field equations which equate the properties of matter with the curvature of spacetime.

The equivalence of the inertial and gravitational masses arises directly form the existence of matter as Mills orbitals. The present derivation of the General Theory of Relativity is based on this principle and the principle that all particles including light follow geodesics. Then the Mills Theory of Quantum Gravitation is given by the solution of the gravitational field equations of the General Theory of Relativity described in Appendix VII.

According to the novel atomic model of the present invention, the energy of a vacuum is zero; thus, the cosmological constant of the model is zero as demonstrated in Appendix VI. This is the exact experimentally determined value. The present novel atomic model, unifies atomic theory and gravitation on a cosmological scale. It is demonstrated in Appendix VII that the solution of Einstein's field equations as a boundary value problem of a Mills orbital unifies atomic theory and the General Theory of Relativity on an atomic level to provide a completely unified gravitational theory for spacetime of all scales. The application of the relevant equations are provided in Appendices VI and VII.

The solution of the gravitational field equation given in Appendix VII permits a result of the opposite sign. The positive result for α of equation (57.37) of Appendix VI arises from positive curvature, and a negative result arises form negative curvature of matter. Thus, antigravity can be created by forcing matter as Mills orbitals, into negative curvature. A fundamental particle with negative curvature would experience a central but repulsive force with a gravitating body comprised of matter of positive curvature. The antigravity force provides a mechanical force; thus, production of matter with negative curvature is the basis of a propulsive means. The propulsive means comprises a source of fundamental particles such as electrons (which are leptons) where the fundamental particles are forced to be essentially plane waves of matter. The plane wave nature of matter results as a Mills orbital absorbs energy to cause the radius to go to infinity as described in the previous patent application. The plane waves of matter are accelerated and formed (or warped) into negative curvature by one or more of an electric field, a magnetic field or an electromagnetic field such as a laser beam applied parallel or transversely to the plane wave of matter or such as an evanescent field produced by a totally.internally refracted electromagnetic wave traveling in a fiberoptic cable.

The antigravity force which arises is transferred to the source means of the fields and is further transferred to the structure to be accelerated or levitated due to the latter means rigid attachment to the structure.

According to the present invention, the force generated by the antigravity levitation/propulsion means can be calculated rigorously by solving Einstein's field equations as a boundary value problem for a three-dimensional spacetime manifold of negative curvature which is produced by the apparatus. However, forces in the limit can be obtained as follows. Consider a negative solution to the variable α of equation (57.37). The negative solution arises naturally as a match to the boundary condition of matter with negative curvature. Furthermore, matter having negative curvature would occupy a diminished quantity of four-dimensional spacetime, as compared to matter of positive curvature. The surface to volume ratio of a sphere is a minimum. In effect μ of equation (57.38) would increase. Consequently, the integral of equation (57.37) is approximately of the form

$$\frac{m4\pi\rho^2}{s} \qquad (1.9)$$

where s is the space defined by the boundaries of he matter of negative curvature. The presence of a three-dimensional spacetime manifold (spatial two-dimensional manifold) in four-dimensional spacetime results in curved nonlocal spacetime which is the origin of gravity, and a two-dimensional spacetime manifold (spatial one-dimensional manifold) in four-dimensional spacetime results in infinite gravity, a singularity which is a black hole. The sign of the curvature. For the case of negative curvature, the antigravity force with a gravitating body can be increased by increasing the intensity of negative curvature.

Further according to the present invention, negatively curved matter is created by "ionizing" fundamental particles to become plane waves. As described in the previous applications, Mills orbitals can absorb electromagnetic energy. As photons are absorbed the radius of a Mills orbital expands from the ground state with radius r_0 to $n\,r_0$ where $n = 2, 3, 4, ...\infty$. As n goes to infinity, the radius r goes to infinity, and the Mills orbital becomes a plane wave. Ionization occurs when sufficient electromagnetic energy has been absorbed to produce a plane wave of a Mills orbital. The ionization energy can be provided by applying a large potential to or by heating or irradiating a cathode. In the latter case, photocathodes irradiated with continuous wave or pulsed lasers can generate very bright, high current density beams of electrons. Photocathodes, thermionic cathodes, and cold cathodes are described by Orttinger, P., et al., Nuclear Instruments and Methods in Physics Research, A272, 264-267 (1988) and Sheffield, R., et al., ibid, 222-226 which are incorporated herein by reference. The resulting plane waves are caused to propagate through space and to acquire negative curvature by traversing a selected field as created by a field source means. The field source means provides one or more of an electric field, a magnetic field, or an electromagnetic field. The resulting spacetime manifold is three-dimensional (spatial two-dimensional manifold and time dependent) and is a solution to the three-dimensional wave equation that follows:

$$(\nabla^2 + \frac{1}{V^2}\frac{\delta^2}{\delta t^2})\,A(x, y, z, t) = 0 \qquad (1.10)$$

Furthermore, the manifold propagates through space and is decelerated by the antigravity force with a gravitating body and is accelerated by the propagation force provided by the source means. The resulting spacetime manifold of negative curvature which arises from the forces acting on the matter is such that its spacetime Fourier transform does not possess waves synchronous with those traveling at the speed of light. The manifold is a Mills orbital which provides an antigravitational force.

Matter (as a Mills orbital) of negative curvature which moves at constant velocity has a spacetime Fourier transform which does not possess Fourier components synchronous with waves traveling at the speed of light. Consider the three-dimensional spacetime manifold

$$\delta\,[z - f(x)\,g(y) - K(t)] \qquad (1.11)$$

where K(t) = Vt; V is a constant.

The spacetime Fourier transform is given as follows:

$$F(k_x)\, G(k_y)\, \delta\, (w - k \cdot \bar{v}) \qquad (1.12)$$

where $F(k_x)$ and $G(k_y)$ is the Fourier transform of f(x) and g(y), respectively.

The only nonzero Fourier components are for

$$k = \frac{w}{v \cos\theta} > \frac{w}{c}$$

where θ is the angle between V and K. Thus, the spacetime Fourier transform has no components synchronous with waves at the speed of light; therefore, the manifold is nonradiative.

For example, the Fourier transform of the three-dimensional spacetime manifold δ (z - xy - vt) is given as follows:

$$\frac{\pi/2}{k_z} e^{-k_x k_y / k_z}\, \delta(w - k \cdot \bar{v})$$

which has no components synchronous with waves traveling at the speed of light; thus, it is nonradiative.

In a preferred embodiment the manifold is given by the following function

$$\delta\, [z - x(z)\, y(z) - v_z t]$$

where v_z is constant velocity in the z direction at force balance. The manifold is produced by a quadrapole electric field at infinity or a quadrapole magnetic field at infinity, and a constant force of equal magnitude and opposite direction of the antigravity force; thus, the matter of negative curvature moves with constant velocity v_z.

THE EMBODIMENT

In one embodiment according to the present invention, the apparatus for providing the antigravitational force comprises a means to inject electron plane waves and a guide means to guide the propagation of the plane waves. Acceleration and forming negative curvature is effected in the propagating guided electrons by application of one or more of an electric field, a magnetic field, or an electromagnetic field by a field source means. A repulsive force of interaction is created between the propagating electrons of negative curvature and the gravitational field of a gravitating body which comprises matter of positive curvature where the field source means provides an equal and opposite force to the repulsive force. Thus, the interactive force is transferred to the field source and the guide which further transfers the force to the attached structure to be accelerated.

In the embodiment, the antigravity means shown schematically in Fig. 4 comprises an electron beam source 100, and an electron accelerator module 101, such as an electron gun, an electron storage ring, a radiofrequency linac, an introduction linac, an electrostatic accelerator, or a microtron. The beam is focused by focusing means 112, such as a magnetic or electrostatic lens, a solenoid, a quadrapole magnet, or a laser beam. The electron beam 113, is directed into a channel of electron guide 109, by beam directing means 102 and 103, such as dipole magnets. Channel 109, comprises a field generating means to produce a constant electric or magnetic force in the direction opposite to direction of the antigravity force. For example, given that the antigravity force is negative z directed as shown in Fig. 4, the field generating means 109, provides a constant z directed electric force due to a constant electric field in the negative z direction via a linear potential provided by grid electrodes 108 and 128; given that the antigravity force is positive y directed as shown in Fig. 4, the field generating means 109, provides a constant negative y directed electric force due to a constant electric field in the negative y direction via a linear potential provided by the top electrode 120, and bottom electrode 121, of field generating means 109. Given that the antigravity force is positive y directed, the field generating means 109, provides a constant negative y directed magnetic force due to a constant dipole magnetic field in the x direction for an electron beam traveling in the z direction.

In one embodiment the field generating means 109, further provides an electric or magnetic field at infinity which warps the electrons of the electron beam 113, into negative curvature to produce the antigravitational force with a gravitating body. In a preferred embodiment the electric potential of the warping electric field is given as follows:

xyz + cp where c is a constant and p is either x, y, or, z and is the direction opposite the force of antigravitation; so, the corresponding electric force on the electron is opposite the antigravitational force as described previously. The electric field is given by the negative of the gradient of the potential. The electric warping force in the z direction is shown in Fig. 2.

In a preferred embodiment the magnetic potential of the warping field is given as follows:

xy + cp where c is a constant and p is either x, y, or z so that the corresponding constant dipole magnetic field produces a constant magnetic force in the direction opposite to the force of antigravity as described previously. The potential function and field lines are shown in Fig. 1. The magnetic field is given by the negative gradient of the potential. The z directed warping force on an electric plane wave propagating in the positive z direction is shown in Fig. 2.

The electric and magnetic warping fields force the electron plane wave into the manifold of negative curvature given as follows:

$$\delta\,[z - x(z)\,y(z) - vt]$$

This manifold is shown schematically in Fig. 3.

The velocity, V, of the manifold is a constant due to the equality of the constant electric or magnetic force and the antigravitational force which arises as an interaction between the gravitating body and the manifold of negative curvature. The constant force provides constant levitation or propagation work against the gravitational field of the gravitating body as the manifold propagates along the channel of the guide means and field producing means 109. The resulting work is transferred to the means to be propelled or levitated via its attachment to field producing means 109.

The constant electric or magnetic force is variable until force balance with the antigravitational force is achieved. In the absence of force balance, the electrons will be accelerated and the emittance of the beam will increase. Also, the accelerated electrons will radiate; thus, the drop in emittance and/or the absence of radiation is the signal that force balance is achieved. The emittance and/or radiation is detected by sensor means 130, such as a photomultiplier tube, and the signal is used in a feedback mode by analyzer-controller 140 which varies the constant electric or magnetic force by controlling the potential or dipole magnets of (field producing) means 109 to control force balance to maximize antigravitational work.

In another embodiment the negative curvature of the electrons of the electron beam 113 is produced by the absorption of photons provided by a photon source 105, such as a high intensity photon source, such as a laser. The laser radiation can be confined to a resonator cavity by mirrors 106 and 107.

In a preferred embodiment the laser radiation or the resonator cavity is oriented relative to the propagation direction of the electron plane wave so that the selection rule angular relationship for the quadrapole transition with zero change in angular momentum is maximized for radiation of a given multipolarity. For example, given (that) the direction of propagation of the beam 113 is in the z direction of Fig. 4, and the radiation is of multipolarity M1 (magnetic dipole radiation), the orientation of the laser radiation or resonator cavity is along the x or y axis (i.e., 90° to the electron beam); given the direction of propagation of the beam 113 is in the z direction, and the radiation is of multipolarity E_2 (electric quadrapole radiation), the orientation of the laser radiation or resonator cavity is along the z axis (i.e., 0° to the orientation of the electron beam).

Following the propagation through field generating means 109 in which antigravity work is extracted from the beam 113, the beam 113, is directed by beam directing apparatus 104, such as a dipole magnet into electron-beam dump 110.

In a preferred embodiment, the beam dump 110 is replaced by a means to recover the remaining energy of the beam 113 such as a means to recirculate the beam or recover its energy by electrostatic deceleration or deceleration in a radio frequency-excited linear accelerator structure. These means are described by Feldman, D. W., et al., Nuclear Instruments and Methods in Physics Research, A259, 26-30 (1987) which is incorporated by reference.

The present invention comprises high current and high energy beams and related systems of free electron lasers. Such systems are described in the following references which are incorporated herein by reference:

Nuclear Instruments and Methods in Physics Research, A272, (1,2), 1-616 (1988)

Nuclear Instruments and Methods in Physics Research, A259, (1,2), 1-316 (1987)

In atoms or in free space, Mills orbitals satisfy force balance. Thus, an electron as a plane wave is accelerated by the force of an electric field, and a nonradiative Mills orbital of negative curvature moves at constant velocity and exists when the forces of absorbed photons, shaping/warping forces, the propagation acceleration forces, and the repulsive gravitational force between the Mills orbital and a gravitating body comprising matter of opposite (positive) curvature exactly balance. The Mills orbital does constant antigravity work as it propagates along the guide where the gravitating body's and the Mills orbital's curvatures are essentially constant over the time of interaction of the gravitational forces.

For a propagation electric field strength of 10^9V/m and a gravitational interaction of 1 meter, the antigravity work of the electron is 1 GeV.

The propulsion power available for guide or a series of guides (109 of Fig. 4) carrying a total of 1000 Amps with a repulsive gravitational interaction force-distance product per electron of 1 GeV is given as follows:

$$\frac{10^9 ev}{electron} \times 1.6(10)^{-19} J/ev \times 1000\ c/sec \times \frac{1\ electron}{1.6(10)^{-19}\ c} =$$

$$10^{12}\ J/sec = \text{one terawatt}$$

The time to accelerate a structure such as a vehicle having a mass of 500,000 kg to a velocity of 1000 m/sec is given as follows:

$$\frac{1}{2} \times 500{,}000 \times (1000\ m/sec)^2 = 2.5 \times 10^{11}\ J$$

$$\frac{2.5 \times 10^{11}}{10^{12}\ J/sec} = .250 \text{ seconds} = 250 \text{ milliseconds}$$

Thus, the antigravity force produced by the antigravity apparatus according to the present invention can be applied to accelerate large vehicles or to levitate any large object.

In a further embodiment, the force provided by the antigravity apparatus according to the present invention is central with respect to the gravitating body. However, acceleration in a direction tangential to the gravitating body's surface can be effected via conservation of angular momentum. Thus, an accelerated structure such as an aerospace vehicle to be tangentially accelerated possess a cylindrically or spherically symmetrically movable mass having a moment of inertia, such as a flywheel device. The flywheel is driven with angular motion by a driving device which is powered and energized by an electric motor and an electric energy source means such as a fusion reaction with a thermionic or steam generator, or batteries. The driving device provides angular momentum to the flywheel. The vehicle is levitated using antigravity means to overcome the gravitational force of the gravitating body where the levitation is such that the angular momentum vector of the flywheel is parallel to the central vector of the gravitational force of the gravitating body.

The angular momentum vector of the flywheel is forced to make a finite angle with the central vector of gravitational force by tuning the symmetry of the levitating (antigravitational) forces provided by antigravity apparatus. A torque is produced on the flywheel as the angular momentum vector is reoriented with respect to the said central vector due to the interaction of the central force of gravity of the gravitating body, the force of antigravity of the antigravity means, and the angular momentum of the flywheel device. The resulting acceleration which conserves angular momentum is perpendicular to the plane formed by the central vector and the angular momentum vector. Thus, the resulting acceleration is tangential to the surface of the gravitating body.

The equation that describes the motion of the vehicle with a moment of inertia I, a spin, moment of inertial I_S, a total mass m, and a spin frequency of its flywheel device of S is given as follows:

$$S = \frac{mgl}{I_S \dot{\phi}} + \frac{I}{I_S} \dot{\phi} \cos \theta$$

$$\dot{\phi} \simeq \frac{mgl}{I_S S} \simeq \frac{mgl}{mr^2 S} = \frac{gl}{r^2 S}$$

where θ is the tilt angle between the central vector and the angular momentum vector, g is the acceleration due to gravity of the gravitating body, l is the height to which the vehicle levitates, and $\dot{\phi}$ is the angular procession frequency resulting from the said torque. The schematic appears in Fig. 5.

A calculation of the approximate velocity achieved when the vehicle's angular momentum vector is tilted 45° with respect to the central vector is given as follows where g = 10 m/sec, l = 5000 m, r = 10 m, S = 25 sec^{-1}

$$\dot{\phi} \cong \frac{gl}{Sr^2} = \frac{(10)(5000)}{(25)(10)^2} = \frac{20 \text{ cycles}}{\text{second}}$$

The linear velocity is the radius times the angular frequency which is given as follows:

$$2\pi\, 20 \text{ cycles/second } (5000\text{m})\sin(45°) = 4.4 \times 10^5 \text{ m/sec}$$

This calculation indicates that large tangential velocities are achievable by executing a trajectory which is vertical followed by tangential (velocities) where the latter motion is effected by tilting the flywheel. During the tangential acceleration energy stored in the flywheel is converted to kinetic energy of the vehicle. The equation for rotational kinetic energy E_R and transitional kinetic energy E_T are given as follows:

$E_R = 1/2I\, \omega^2$ where I is the moment of inertia and w is the angular rotational frequency;

$E_T = 1/2\, mv^2$ where m is the total mass and V is the transitional velocity.

The equation for the moment of inertia I of the flywheel is given as:

$I = \Sigma\, mr^2$ where m is the infinitesimal mass at a distance r from the center of mass. These equations demonstrate that maximum rotational kinetic energy can be stored for a given mass by maximizing the distance of the mass from the center of mass. Thus, ideal design parameters are cylindrical symmetry with the rotating mass at the perimeter of the vehicle.

Furthermore, according to the methods and apparatus of the present invention providing antigravitational forces, rapid long distance transport may be realized where the propelled means, such as a space vehicle, is accelerated to enormous velocity by executing a hyperbolic trajectory around a gravitating body wherein the force of gravity of the gravitating body and the antigravity force of the vehicle provided by the antigravity means of the present invention accelerate the vehicle to high velocity.

EXPERIMENTAL

A high current, high energy electron beam was injected into a quadrapole magnetic field, and the geometric cross-sectional profile of the beam was recorded by Carlsten (Carlsten, B. E.; et al., Nuclear Instruments and Methods in Physics Research, A272, 247-256 (1988). One embodiment of the antigravity propulsion and levitation means of the present invention comprises the apparatus of Fig. 6 with the absence of the wiggler and the spectrometer. But, in addition the device of the present invention comprises an electron guide means comprising a channel for the electron beam and a field generating means 109 of Fig. 4, to produce a constant electric or magnetic force against the antigravitational force produced on the electrons of negative curvature following their propagation through the quadrapole triplets, Q_1, Q_2, and Q_3 of Fig. 6. Unharnessed antigravity was achieved as demonstrated by the flame shape of the beam which is a function of current as shown in Fig 7. (which is Fig. 11 of the reference). The data indicate that a Boltzmann distribution of negative curvature was achieved as is apparent by the flame shape of the beam profile (see Fig. 7). The shape is due to the constant gravitational field of the Earth interacting with a Boltzmann distribution of manifolds of negative curvature resulting in a Boltzmann distribution of antigravitational forces and corresponding displacements. The maximum vertical deflection of the relativistic electrons by the antigravitational forces is approximately 5 centimeters over a displacement in the direction of the electron beam of 50 centimeters. Thus, antigravitational forces comparable to the electrostatic and electromagnetic forces of the apparatus were achieved. The current dependence of the efficiency of negative curvature production resulted from increased electron-electron interactions with higher beam current which prevented efficient coupling of the electrons with the quadrapole triplets. However, significant antigravity was produced at currents of several hundred amperes. Thus, the present experiment indicates that antigravitational work of the order of 1 GeV per electron is achievable by the methods and apparatus of the present invention.

The present invention unifies the three forces, electromagnetic, gravitational, and mechanical and permits their interconversion. As further examples of structure and methods of the present invention, the Meisner effect is the phenomenon whereby a superconductor of the present invention converts a gravitation force into an electromagnetic force, and the energy of the strong nuclear force is released as heat during Coulombic Annihilation (cold) Fusion of the present invention which produces electricity directly via a thermionic or photovoltaic generator, or the heat produces a mechanical force via steam which turns a generator to create electricity, as illustrated in Fig. 8. The electricity provides an electromagnetic force which by a device of the present invention warps matter into negative curvature such that an antigravitational force is produced. The antigravitational force provides useful propulsion or levitation as a mechanical force.

In one embodiment shown in Fig. 8, the fusion reactor 210 provides heat via Coulombic Annihilation Fusion which is converted to steam in heat exchanger 214. The steam is transferred by connection 216 to turbine 218 which is driven by the steam to produce electricity to supply the electrical load of the antigravity apparatus 224. Alternatively, the heat is transferred by connection 212 to thermionic power converter 226 which directly converts the heat to electricity to supply the electrical load of the antigravity apparatus 224, where the unused heat is returned via connection 213. The electrical energy is converted into antigravitational energy by antigravity apparatus 228 which provides propulsion and levitation to the vehicle to which the antigravity apparatus 228 is structurally attached by structural connection 206. The fusion reactor 210, the heat exchanger 214, the turbine 218, the power generator 220, and the thermionic power converter 226, are also propelled or levitated with the vehicle by their respective structural connections 201-206 to the vehicle.

APPENDIX I

Basic Concepts in Relativistic Astrophysics, Li Zhi Fang and Remo Ruffini, World Scientific Publishing Co. Pte. Ltd.,1983,Chapters 1 and 2, pp 1-70 is incorporated herein by reference.

APPENDIX II

(The following text is published as Chapter V of the Theory of Space, Time and Gravitation, 2nd revised ed, Pergamon Press, pp 228-233 (1965) and is incorporated herein by reference.

According to the present invention, departures from the text have been made. Additions are underlined and deletions are [bracketed] and initialed.)

In the theory of gravitation the Principle of Equivalence is understood to be the statement that in some sense a field of acceleration is equivalent to a gravitational field. The equivalence amounts to the following. By introducing a suitable system of coordinates (which is usually interpreted as an accelerated frame of reference) one can so transform the equations of motion of a mass point in a gravitational field that in this new system they will have the appearance of equations of motion of a free mass point. Thus a gravitational field can, so to speak, be replaced, or rather imitated, by a field of acceleration. Owing to the equality of inertial and gravitational mass such a transformation is the same for any value of the mass of the particle. But it will succeed in its purpose only in an infinitesimal region of space, i.e., it will be strictly local.

In the general case the transformation described corresponds mathematically to passing to a locally geodesic system of coordinates. As was shown by Fermi, it is possible to introduce coordinate systems which are locally geodesic not only at one point but also along a time-like world-line.

Thus the principle of equivalence is related to the law of equality of inertial and gravitational mass, but is not identical with it. The latter is of a general, non-local character while the equivalence of a field of acceleration and a field of gravitation exists only locally, i.e., it refers only to a single point in space (more precisely to a spatial neighbourhood of the points on a world-line, which is of the nature of a time axis).

The Principle of Equivalence played an important role during the period before Einstein created his theory of gravitation. We shall now describe and analyse an argument given by Einstein at that time.

Einstein illustrated his "equivalence hypothesis" with the example of a laboratory inside a falling lift. All objects within such a lift appear bereft of their weight, they all fall together with the lift, with the same acceleration, so that their relative accelerations vanish even when they are not fixed to the walls of the lift. We have, according to Einstein, two frames of reference, one inertial, or almost inertial, fixed to the Earth and another accelerated, fixed to the lift. In the first, inertial frame, there exists a gravitational field - in the second, accelerated frame, it is absent. Thus, according to Einstein, an acceleration can replace gravitation or at least a uniform field of gravitation. Einstein develops this idea further. He proposes to consider both the accelerated and the unaccelerated frames to be physically completely equivalent and points out that from such a point of view the concepts of inertial frame and absolute acceleration cease to have any meaning.

Let us analyse this view of Einstein's in more detail. First of all the question arises: What is an accelerated frame of reference and how can it be realized physically? In the lift example the "frame of reference" was, so to speak, identified with a certain box, the lift cage. But we have [ ] learned that even when gravitation is not taken into account the abstraction of an absolutely rigid body is not acceptable; when accelerated all bodies will experience deformations which will be different for different bodies[.] where the effects are given by special relativity. Therefore a box or a rigid scaffolding [of the kind we discussed in Section 11 when dealing with inertial frames are] is of no use as a model[s] for an accelerated frame of reference. Thus in Einstein's reasoning the basic concept of a frame of reference in accelerated motion remains undefined. This difficulty could be avoided only by imposing limitations on the magnitude of the acceleration and on the size of the region of space to be considered. For instance, one could demand the following: the accelerations allowed are to be so small that in the region of space considered the deformations resulting from them may be neglected and the notion of a rigid body may be used. In that case the approximate nature of Einstein's argument becomes obvious.

Further, Einstein himself stresses that not every gravitational field can be replaced by acceleration; for this to be possible the gravitational field must be uniform. This also imposes limitations on the spatial dimensions of the region in which gravitational and accelerated fields may be approximately equivalent. It is, for instance, impossible to "remove" the gravitational field around the terrestrial globe; to do it one would have to introduce some absurdity such as a frame of reference in "accelerated contraction."

Einstein also used his Principle of Equivalence in a non-local manner but his attempt, in a paper published in 1911, to investigate in this way the propagation of light near a heavy body gave a deflection of a light ray of only half the amount resulting from his theory of gravitation [see Section 59.] as calculated in Appendix I. This is connected with the fact that the Principle of Equivalence cannot possibly lead to the correct form (51.11) for ds^2 but at best only the expression (51.10) which is valid for slow motion[.] where ds^2 is the infinitesimal displacement in an inertial frame which is derived in Appendix VI. Thus, in a non-local interpretation, the approximate equivalence of fields of gravitation and of acceleration is also limited. As already mentioned this equivalence exists only for weak uniform fields and slow motions.

Einstein gave to his principle of equivalence a widened interpretation by taking it to imply the indistinguishability of fields of gravitation and acceleration and asserting that from the point of view of this principle it is as impermissible to speak of absolute acceleration as it is to speak of absolute velocity. To this Einstein related his "General Principle of Relativity" [which we discussed in Section 49*] (Appendix V); he used the latter to justify the demand that his equations should be generally covariant. (The concept of covariance is discussed in Appendix V.) However, [to us] such an extended interpretation seems inconsistent. The essence of the principle of equivalence may be seen in the fact that it allows the introduction of an appropriate locally geodesic ("freely falling") frame of reference, by use of which a uniform Galilean space can be defined in the infinitesimal. However this in no way justifies conclusions about the equivalence or indistinguishability of fields of acceleration and of gravitation in finite regions of space. To illustrate the nature of the error committed in drawing such conclusions let us examine a mathematical example, which incidentally has a direct bearing on the essence of the present question. All functions that have bounded second derivatives behave as linear functions in the infinitesimal. However, this by no means allows one to conclude that all such functions are indistinguishable in a finite region. But an analogous conclusion, namely that fields of acceleration and of gravitation are completely indistinguishable, was drawn by Einstein, on the basis of their local equivalence alone.

Such a conclusion even contradicts Einstein's theory of gravitation itself. Indeed, if full equivalence between fields of acceleration and of gravitation did exist, a theory built on the idea of equivalence would be purely kinematical, which is by no means the case for Einstein's theory of gravitation. As regards the "General Principle of Relativity," [we have already pointed out in Section 49*] it is demonstrated in Appendix V that such a physical principle is impossible, and also unnecessary as a basis for the requirement of general covariance, which is the purely logical requirement of consistency for a theory in which the coordinate system is not fixed.

Thus, although the principle of equivalence holds in a narrow sense (approximately and locally) it does not hold in a wider sense. Although the effects of acceleration and of gravitation may be indistinguishable "in the small," i.e., locally, they are undoubtedly distinguishable "in the large," i.e. when the boundary conditions to be imposed on gravitational fields, are taken into account. The gravitational potential that is obtained if a uniformly accelerated frame of reference in introduced is a linear function of the coordinates and therefore does not satisfy the conditions at infinity, where it should tend to zero. In a rotating frame of reference the potential of the centrifugal force increases with the square of the distance from the axis of rotation, and in addition there are Coriolis forces. By these characteristics it is possible to detect immediately that the "gravitational field" in such frames of reference is fictitious.

We shall now discuss the example of a uniformly accelerated frame of reference in somewhat greater detail, taking the theory of relativity into account. In doing this we set aside the question of how an accelerated frame might be realized and interpret the term "frame of reference" more formally in the sense of "coordinate system." In this sense passing to a frame moving with acceleration will mean subjecting the coordinates to a transformation which contains time non-linearly.

We assume that a true gravitational field is not present so that the square of the infinitesimal interval has the form

$$ds^2 = c^2dt'^2 - (dx'^2 + dy'^2 + dz'^2) \qquad (61.01)$$

where x', y' and z' are Cartesian coordinates and t' the time in some inertial frame of reference. We perform the coordinate transformation[s]

$$x' = x \cosh \frac{gt}{c} + \frac{c^2}{g}\left(\cosh \frac{gt}{c} - 1\right)$$
$$y' = y; \; z' = z \qquad (61.02)$$
$$t' = \frac{c}{g} \sinh \frac{gt}{c} + \frac{x}{c} \sinh \frac{gt}{c}$$

where g is a constant of the dimensions of acceleration. Under the condition

$$\frac{gt}{c} << 1 \qquad (61.03)$$

the previous equations may be written as

$$x' = x + \frac{1}{2} gt^2; \quad y' = y; \quad z' = z; \quad t' = t \qquad (61.04)$$

Inserting (61.02) into (61.01) we obtain

$$ds^2 = \left(c + \frac{gx}{c}\right)^2 dt^2 - (dx^2 + dy^2 + dz^2) \qquad (61.05)$$

The question arises: can this expression be interpreted as the square of the interval in some inertial frame of reference in which there is a gravitational field? The answer to this question is also an answer to the question whether, and to what extent, the Principle of Equivalence is correct.

To find the answer we compare (61.05) with the approximate expression given by the theory of gravitation.

$$ds^2 = (c^2 - 2U)\, dt^2 - \left(1 + \frac{2U}{c^2}\right)(dx^2 + dy^2 + dz^2) \qquad (61.06)$$

where U is the Newtonian potential of a true gravitational field.

Under the condition

$$|gx| << c^2 \qquad (61.07)$$

the coefficients of dt^2 are approximately equal if we take a gravitational potential given by

$$U = -gx \qquad (61.08)$$

As for the coefficient of the spatial part of ds^2, it will not differ significantly from unity for intervals for which the quantity

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \qquad (61.09)$$

satisfies the inequality

$$v^2 << c^2 \qquad (61.08)$$

The value (61.08) for the gravitational potential does indeed lead to uniformly accelerated motion in Newtonian mechanics. For vanishing initial velocity we have constant values of x', y' and z' and approximately

$$x + \frac{1}{2} gt^2 = \text{const.} \qquad (61.11)$$

[†] [This transformation was given by Moller [20].] which describes uniformly accelerated motion in the coordinates (x, t).

We have made a comparison between two expressions for the square of the interval which has shown that a frame of reference in accelerated motion in the absence of gravitation does indeed show an analogy with an inertial frame in the presence of gravitation. However, the same comparison indicates that the analogy is far from complete, so that there can be no question of full equivalence or indistinguishability of inertial and gravitational fields. This becomes particularly clear if one considers the expression (61.05) "in the large," i.e. throughout the whole of space. In the first place, the coefficient of dt^2 does not [saitsfy] satisfy the boundary conditions, since it tends to infinity with x, in the second place that coefficient and with it the speed of light, become zero on the surface $x = -c^2/g$; this is inadmissible.

An even more obvious violation of the boundary conditions for the metric tensor occurs if the transformation (35.47) is used. In Newtonian mechanics it has the significance of introducing a rotating coordinate system. This transformation leads to the expression (35.48) for ds^2 [.]) (35.47) and (35.48) appear in Appendix IV.) Here the metric tensor not only fails to satisfy the boundary conditions but, at large distances from the axis of rotation, also violates the inequalities established in [Section 35] Appendix IV. The impossibility of interpreting the metric tensor in (35.48) as some gravitational field (i.e. in the spirit of the "equivalence hypothesis") is clear even from a local point of view, owing to the presence of Coriolis forces.

The example just discussed confirms completely the conclusion stated above that the "equivalence" between acceleration and gravitation exists only in a limited region of space, and only for weak and uniform fields and slow motions (equation (61.08) together with the inequalities (61.07) and (61.10)). But if one considers the whole of space, true gravitational fields can be distinguished from fictitious ones caused by acceleration. In Newtonian theory this can be done by using the boundary condition for the Newtonian potential. In Einstein's theory the question of distinguishing true from fictitious gravitational fields is most simply solved if harmonic coordinates are used. Then the components of the metric tensor must satisfy both the harmonic conditions (53.13) and the boundary conditions discussed [in Section 54] in Appendix VI. [As will be shown in Section 93,] Furthermore, harmonic coordinates can be defined uniquely apart from a Lorentz transformation. Arbitrary coordinate transformations by which fictitious gravitational fields are introduced, violate the harmonic conditions and the boundary conditions. Therefore one can take it that the introduction of harmonic coordinates exclude all fictitious gravitational fields. Thus, if one assumes the quadratic form (61.05) to be given, the passage to harmonic coordinates will consist in the transformation (61.02), accompanied possibly by a Lorentz transformation. As the result of such a transition we come back to the quadratic form (61.01), the form of which indicates the absence of true gravitational fields.

In [the] this discussion [of this section] we did not use general tensor analysis. Its application to (61.05) would have shown that the fourth rank curvature tensor vanishes and that, therefore, true gravitational fields are indeed absent.

Let us return to the question of utilizing the principle of equivalence to derive the gravitational equations. We have made it clear that it is inconsistent to interpret this principle in a wider sense as a "General Principle of Relativity." But this does not exclude the use of the principle of equivalence in a more restricted sense, within the limits in which it is valid approximately. In particular the analogy we have discussed between an accelerated frame of reference in the absence of a gravitational field and an inertial frame in the presence of such a field may prove helpful, because the possibility of transforming the expression (61.01) into the form (61.05) gives us an indication of the fact that the Newtonian potential must enter the theory precisely as the coefficient of dt^2. However, an approach based on this idea to the formulation of a gravitational theory seems [to us] to be unsatisfactory because of its inherent limitations (viz. the local nature of the principle of equivalence and the assumption that the field is uniform). Another disadvantage of this approach is the necessity of using the ill-defined concept of a frame of reference in accelerated motion. [Our] The approach of the present invention is free from these disadvantages, being based on the direct application of the law of equality of inertial and gravitational masses [.] with the solution of Einstein's field equations as a boundary value problem where a discontinuity of matter is equated to a discontinuity of the curvature of spacetime. [It is well to remember that in] In the derivation of Einstein's gravitational equations in Appendices VI and VII, [we made] no use is made of any frame of reference in accelerated motion and therefore also no use of the principle of equivalence. As for this latter principle, to the extent that it is valid it may be obtained subsequently as a consequence of the other assumptions. Thus it is implied by the hypothesis that spacetime has Riemannian character, its mathematical expression being the possibility of introducing a locally geodesic coordinate system along a time-like world-line.

We stressed the approximate character of the principle of equivalence. But from the point of view of Einstein's theory of gravitation the law of equality of inertial and gravitational mass also is of approximate character, since the very concepts of inertial and gravitational mass are approximate. These concepts are applicable to the extent to which Newton's laws of motion and law of gravitation are valid and to the extent that it is possible to define any mass as a quantity characterizing a particular body independently of its position and of the motion of other bodies. In Einstein's theory of gravitation this is possible only approximately, because there the law of motion of material bodies is of a more complicated nature. Nevertheless, one can affirm that the law of equality of inertial and gravitational mass agrees fully with Einstein's theory of gravitation, because this law follows from the theory with as much precision as can in general be given to its formulation.

On the other hand Einstein's theory of gravitation does not reduce to a formulation of the law of equality of the masses; it embraces essential new physical principles. The first is already contained in the ordinary theory of relativity: the unification of space and time into a single four-dimensional manifold with an indefinite metric. This principle is related to the limiting nature of the velocity of light and, closely connected with this, to the more precise specification of what is meant by a sequence of events in time and also by cause and effect [(Section 12)]. The second principle establishes the unity of metric and gravitation; it is the very essence of Einstein's gravitational theory.

It is just these two principles, and not any widening of the concept of relativity, supposedly possible as a result of the local equivalence of acceleration and gravitation, which form the basis of Einstein's theory of gravitation. But they provide no atomic basis for curved spacetime or the equivalence of gravitational of inertial masses.

APPENDIX III

(The following text is published in Nature, Vol. 336, pp. 711-712 (1988), and is incorporated herein by reference.

According to the present invention, departures from the text have been made. Additions are underlined and deletions are [bracketed] and initialed.)

New work by Sidney Coleman [ ] extends Hawking's result by assuming the existence and importance of quantum fluctuations which change the topological structure of spacetime [(see figure)]. It should be stressed that we really have no idea whether such fluctuations can occur, but if they do and if their effects are relevant we can proceed to analyse what those effects might be. The first thing we know is that the connecting wormholes, filaments of distorted spacetime, would have to be very tiny - about 10^{-33}cm, the natural size for gravitational quantum-mechanical fluctuations. Thus they would not be directly observable. Rather, as has been shown, their effect would be transmitted indirectly through the values of the constants of nature.

The way that wormholes affect our universe depends on the number of 'baby universes' they lead to. [(see figure).]. Rather than using the number of baby universes of type i, Coleman uses a closely related variable α_1. Because the wormholes affect the values of the parameters in physical theories and because wormhole effects are governed by the variables of α_1, all the constants of nature become functions of the α_1. This means that particle masses, the fine-structure constant, the gravitational constant and, of course, the cosmological constant all depend on parameters characterizing the topological structure of space. [ ]

Furthermore, if we can predict anything about the distribution of α_1 values we may learn something about the values of physical parameters like the cosmological constant. Because the α_1s are part of our description of spatial geometry, their probability distribution is determined by the wavefunction of the Universe. Using the techniques of Hawking, Coleman finds that the probability distribution for the α_1 contains a factor which is infinitely peaked at values of these parameters which make the cosmological constant vanish (for small positive cosmological constants, it is proportional to the exponential of the exponential of one over the cosmological constant). Thus, the cosmological constant vanishes because it is infinitely more likely that the constants of nature assume values which make it vanish, than that they do not.

[ ] The Coleman-Hawking programme [ ] avoids a major obstacle which has derailed most other attempts to adjust the cosmological constant to zero. [.] Our universe is filled with matter and radiation which throughout most of its history (presumably up to the present) have completely obscured any effects of a small but non-vanishing cosmological constant. So how can any mechanism determine the constant's value and adjust it to zero? The answer in Coleman's approach is that the Universe peeks through a wormhole into a large empty universe thus escaping the problem of the obscuring matter and radiation in our Universe.

APPENDIX IV

(The following text is published in The Theory of Space, Time and Gravitation, Chapter III, pp. 114-135 and is incorporated herein by reference.

According to the present invention, departures from the text have been made. Additions are underlined and deletions are [bracketed] and initialed.)

GENERAL TENSOR ANALYSIS

35. Permissible Transformations for Space and Time Coordinates

As the basis of our mathematical formulation of Relativity Theory we chose the wave front equation

$$(\nabla\omega)^2 \equiv \frac{1}{c^2}\left(\frac{\delta\omega}{\delta t}\right)^2 - \left[\left(\frac{\delta\omega}{\delta x}\right)^2 + \left(\frac{\delta\omega}{\delta y}\right)^2 + \left(\frac{\delta\omega}{\delta z}\right)^2\right] = 0 \qquad (35.01)$$

and the related expression for the square of the interval

$$ds^2 = c^2dt'^2 - (dx^2 + dy^2 + dz^2) \qquad (35.02)$$

($(\nabla\omega)^2$ is to be understood as an abbreviation for the differential expression on the left-hand side of the wave front equation.) If we introduce our usual variables

$$x_0 = ct;\quad x_1 = x;\quad x_2 = y;\quad x_3 = z \qquad (35.03)$$

and also the numbers

$$e_0 = 1;\quad e_1 = e_2 = e_3 = -1 \qquad (35.04)$$

the expressions $(\nabla\omega)^2$ and ds^2 can be written as

$$(\nabla\omega)^2 = \sum_{k=0}^{3} e_k\left(\frac{\delta\omega}{\delta x_k}\right)^2 \qquad (35.05)$$

and

$$ds^2 = \sum_{k=0}^{3} e_k\,(dx_k)^2 \qquad (35.06)$$

We know that both these expressions are invariant under Lorentz transformations. If new coordinates

$$x'_0 = ct';\quad x'_1 = x';\quad x'_2 = y';\quad x'_3 = z' \qquad (35.07)$$

are introduced which are connected with the previous ones by a Lorentz transformation, we get

$$(\nabla\omega)^2 = \sum_{k=0}^{3} e_k\left(\frac{\delta\omega}{\delta x'_k}\right)^2 \qquad (35.08)$$

and

$$ds^2 = \sum_{k=0}^{3} e_k\,(dx'_k)^2 \qquad (35.09)$$

Variables such as (35.03) or (35.07) in which $(\nabla\omega)^2$ and ds^2 have the forms (35.05) and (35.06) or (30.08) and (30.09) will be called Galilean coordinates, this term now being understood to include the time.

We now assume that while x'_0, x'_1, x'_2, and x'_3 are given by (35.07) as before, and so are Galilean coordinates, the quantities $x0$, $x1$, $x2$, and $x3$ are no longer equal to (35.03) but instead are some auxiliary variables connected to x'_0, x'_1, x'_2, and x'_3 by relations of the form

$$x'_\alpha = f_\alpha(x_0, x_1, x_2, x_3) \quad (\alpha = 0, 1, 2, 3) \qquad (35.10)$$

where the f_α are arbitrary functions subject only to some general conditions. We shall assume that the equations (35.10) can be solved for x0, x1, x2, and x3 so that their Jacobian must be non-zero

$$D = \frac{D(x'_0, x'_1, x'_2, x'_3)}{D(x_0, x_1, x_2, x_3)} \neq 0 \qquad (35.11)$$

Further we suppose that the functions f_α have continuous derivatives of the first three orders. There will be other conditions on the f_α which arise from the physical considerations; these will be examined later.

If this change of variables if made $(\nabla\omega)^2$ becomes a homogeneous quadratic form in the first derivatives with respect to the variables x0, x1, x2, and x3. We write this form as

$$(\nabla\omega)^2 = \sum_{\alpha\beta=0}^{3} g^{\alpha\beta} \frac{\delta\omega}{\delta x_\alpha} \frac{\delta\omega}{\delta x_\beta} \qquad (35.12)$$

where

$$g^{\alpha\beta} = \sum_{k=0}^{3} e_k \frac{\delta x_\alpha}{\delta x'_k} \frac{\delta x'_\beta}{\delta x_k} \qquad (35.13)$$

Similarly, if the change of variables is made in ds^2 the result is

$$ds^2 = \sum_{\alpha\beta=0}^{3} g^{\alpha\beta} dx_\alpha dx_\beta \qquad (35.14)$$

with

$$g_{\alpha\beta} = \sum_{k=0}^{3} ek \left(\frac{\delta x'_k}{\delta x_\alpha} \frac{\delta x'_k}{\delta x_\beta} \right) \qquad (35.15)$$

It is readily deduced from (35.13) and (35.15) that

$$\sum_{\alpha=0}^{3} g_{\mu\alpha} g^{\nu\alpha} = \delta^{\nu}_{\mu} = \begin{cases} 1 & \text{if } \nu = \mu \\ 0 & \text{if } \nu \neq \mu \end{cases} \qquad ((35.16)$$

Hence if g is defined as the determinant

$$g = \mathrm{Det}\, g_{\alpha\beta} \qquad (35.17)$$

the quantities $g^{\alpha\beta}$ will be the first minors of this determinant divided by g itself.

Using the rule of determinant multiplication we get

$$\mathrm{Det}\left(e_k \frac{\delta x'_k}{\delta x_\alpha} \right) \cdot \mathrm{Det}\left(\frac{\delta x'_i}{\delta x_\beta} \right) = \mathrm{Det}\, g_{\alpha\beta} \qquad (35.18)$$

Here the second factor is equal to the Jacobian D of (35.11) and since

$$e_0 e_1 e_2 e_3 \qquad (35.19)$$

the first factor is -D. Consequently

(35.20)

It is useful to restrict the choice of variables x0, x1, x2, and x3 by conditions which ensure that x0, like x'0, is of the nature of a time whereas x1, x2, and x3 are of the nature of spatial coordinates. These conditions must be formulated precisely. As before, we mean by the term "event" a spatial point considered at a particular moment in time; it may be called a "point-instant." We demand that two events having the same values of the spatial coordinate parameters x1, x2, and x3 but different values x_0 *and x_0** for their time parameters shall be [in time sequence in the sense of Section 12] time-like. We know that for time-like events [in time sequence] the squared interval is positive. This must be true in particular for an infinitesimal interval, so if we take the difference $x_0^* - x_0^{**}$ to be infinitesimal and put

$$x^*_0 = x_0; \quad x^{**}_0 = x_0 + dx_0 \qquad (35.21)$$

we must have

$$ds^2 = g_{00} dx^2_0 \qquad (35.22)$$

whence

$$g_{00} > 0 \qquad (35.23)$$

Assume further that we have two events with the same time parameter x0 but different values of the spatial parameters, namely (x1, x2, and x3) and ($x1 + dx_1$, $x2 + dx_2$, x3 + dx3). We require that two such events shall be quasi-simultaneous where quasi-simultaneous events are described as follows:

Let us assume now that the light from one flash does not reach the place of [the other] another flash before the latter occurs.

Then inequalities opposite [to (12.01) and (12.02)] that follow will hold:

$$-\frac{1}{c} \mid r_2 - r_1 \mid < t_2 - t_1 < \frac{1}{c} \mid r_2 - r_1 \mid \qquad (12.05)$$

where r_1, r_2 and t_1, t_2 are the position and time of 1 and 2, respectively. Pairs of events for which the inequality (12.05) is true will be called *quasi-simultaneous*. This name is justified by the fact that in this case the notions "earlier" and "later" become relative ones: one may find $t_2 - t_1 > 0$ in one reference frame and $t_2 - t_1 < 0$ in another. The question as to which flash happened first has now no unique answer.

Quasi-simultaneous events can be characterized by the invariant inequality

$$c^2(t_2 - t_1)^2 - (r_2 - r_1)^2 < 0 \qquad (12.06)$$

which follows from (12.05). The two relations, (12.05) and (12.06), are equivalent and, therefore, (12.05) is also invariant. We shall call the real positive quantity

$$R = \sqrt{[(r_2 - r_1)^2 - c^2(t_2 - t_1)^2]} \qquad (12.07)$$

the *space-like interval* between two quasi-simultaneous events. For quasi-simultaneous events ds^2 is negative, therefore we must have

$$ds^2 = \sum_{ik=1}^{3} g_{ik}\, dx_i dx_k < 0 \qquad (35.24)$$

whatever the values of dx_1, dx_2, and dx_3, provided not all three are zero. It follows that the quadratic form (35.24) must be negative-definite. It is a well-known algebraic fact that the necessary and sufficient conditions for this are the inequalities

$$\begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} < 0 \qquad (35.25)$$

$$\begin{vmatrix} g_{11} & g_{22} \\ g_{21} & g_{22} \end{vmatrix} > 0\,; \quad \begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix} > 0\,; \quad \begin{vmatrix} g_{11} & g_{13} \\ g_{31} & g_{33} \end{vmatrix} > 0 \qquad (35.26)$$

$$g_{11} < 0\,; \quad g_{22} < 0\,; \quad g_{33} < 0 \qquad (35.27)$$

which, incidentally, are not all independent. An independent set of conditions is, for instance (35.25), the first inequality of (35.26) and the first of (35.27).

It is not difficult to show that if all these conditions are imposed on the coefficients $g_{\alpha\beta}$ then, regardless of whether they are given by (35.15) or not, it is possible to represent ds^2 in the neighbourhood of any point as the sum of four squared terms, one with a positive and the remaining three with negative signs. The set of signs of the terms is called the signature of the quadratic form. In our case the signature can be written as (e_0, e_1, e_2, and e_3) or as (+ - - -).

It follows from the inequalities (35.25) to (35.27) that the determinant g is always negative and also that similar inequalities involving the $g^{\alpha\beta}$ with upper indices hold, consequently we have

$$g^{oo} > 0 \qquad (35.28)$$

and

$$\sum_{ik=1}^{3} g^{ik}\,\omega_i\omega_k < 0 \qquad (35.29)$$

where $\omega 1$, $\omega 2$, and $\omega 3$ are any three numbers, not all zero. We shall not give the proofs of these purely algebraic statements.

Thus, in order that the parameter x0 should have the character of time and the other three, x1, x2, and x3, the character of spatial coordinates, it is necessary and sufficient that g00 should be positive and that the quadratic form with the coefficients gik (i, k = 1, 2, 3) should be negative-definite. There is no need to impose any restriction on the quantities g10, g20, and g30.

Let us now consider the geometrical meaning of the equations x0 = const. and xk = const. We shall derive a condition under which the condition

$$\omega(x, y, z, t) = 0 \qquad (35.30)$$

can be interpreted as the equation of a surface in motion. It follows from this equation that the differentials of space and time coordinates, are related by

$$\omega_x dx + \omega_y dy + \omega_z dz + \omega_t dt = 0 \qquad (35.31)$$

where ωx, ωy, ωz, and ωt denote the derivatives of ω with respect to x, y, z and t. We take a displacement (dx, dy, dz) in the direction of the normal to the surface and put

$$dx = \frac{\omega_x}{|\mathrm{grad}\,\omega|}dn; \quad dy = \frac{\omega_y}{|\mathrm{grad}\,\omega|}dn; \quad dz = \frac{\omega_z}{|\mathrm{grad}\,\omega|}dn; \qquad (35.32)$$

so that $|dn|$ is the absolute value of the displacement. Inserting in (35.31) we get

$$|\mathrm{grad}\,\omega|\,dn + \omega_t dt = 0 \qquad (35.33)$$

and therefore the square of the displacement velocity

$$v^2 = \left(\frac{dn}{dt}\right)^2 \qquad (35.34)$$

will be given by

$$v^2 = \frac{\omega_t^2}{|\mathrm{grad}\ \omega|^2} \qquad (35.35)$$

Thus (35.30) can be interpreted as the equation of a surface, each point of which moves normally with a speed given by (35.35). However, such an interpretation is only possible as long as this speed does not exceed that of light. According to (35.35) and (35.01) this means that we must have

$$(\nabla\omega)^2 \leq 0 \qquad (35.36)$$

The equality sign is valid for motions with the speed of light.

On the other hand, is

$$(\nabla\omega)^2 > 0 \qquad (35.37)$$

equation (35.30) can be solved for the time and written in the form

$$t = \frac{1}{c} f(x, y, z) \qquad (35.38)$$

with

$$(\mathrm{grad}\ f)^2 < 1 \qquad (35.39)$$

Equation (35.38) assigns to every point in space a definite instant of time in such a way that all the four-dimensional "point -instants" are quasi-simultaneous. Such an equation may be called a "time-equation." [We recall that] also time equations occur[red] [in Section 3] in connection with the question of the characteristics of Maxwell's equations.

[As we remarked in Section 3, an] An equation $\omega = 0$ can be considered as the equation of a hyperspace in the four-dimensional spacetime manifold. Such hyperspaces can then be divided into two classes.

If $(\nabla\omega)^2 < 0$ we can say that one of the dimensions of the hyperspace is time-like (the inaccurate phrase "the surface is time-like" is sometimes used). By (35.35) this describes an ordinary two-dimensional surface* moving with a velocity less than that of light.

If $(\nabla\omega)^2 > 0$, on the other hand, we say that the hypersurface is space-like. It then represents the whole of infinite space, the various points of which are all taken at different instances of time, the time t at which the point (x, y, z) is taken, being determined by the time equation, i.e. the equation of the hypersurface; the instants of time assigned to any two points in space must be so close that the corresponding four-dimensional interval is always space-like.

We use the fact that $(\nabla\omega)^2$ is an invariant and in turn put $\omega = x_0$, $\omega = x_1$, $\omega = x_2$, and $\omega = x_3$. This gives

---

* In the four-dimensional manifold a hypersurface has three dimensions but in the present case only two of these are spatial.

$$(\nabla x_0)^2 = g^{oo} > 0 \qquad (35.40)$$

and

$$(\nabla x_1)^2 = g^{11} < 0; \;\; (\nabla x_2)^2 = g^{22} < 0; \;\; (\nabla x_3)^2 = g^{33} < 0 \qquad (35.41)$$

Hence the equation x_0 = const. is a time equation and the three equations x_k = const. (k = 1, 2, 3) represent surfaces moving in the direction of their normals with less than light velocity. These latter are thus equations of moving spatial coordinate surfaces.

It follows also from our conditions on the transformations of space and time coordinates that constant values of $x1$, $x2$, and $x3$ correspond, in any inertial frame of reference, to motion of a point with less than light velocity.

In classical Newtonian mechanics one often uses a time dependent coordinate transformation which is interpreted as passing to a moving frame of reference. In comparing coordinate transformations in Newtonian mechanics with the transformations of time and space coordinates in the Theory of Relativity it is essential to realize the following. Firstly, in the general case of accelerated motion the very notion of an accelerated frame of reference in Newtonian mechanics is not the same as in Relativity. The Newtonian concept involves the idea of an absolutely rigid body and the instantaneous propagation of light. In Relativity, on the other hand, the notion of a rigid body is used, if at all, not in an absolute sense but only for non-accelerated motions and in the absence of external forces, and is of an auxiliary nature; the concept of a frame of reference is not based on it but on the law of wavefront propagation. The prototype of a Newtonian frame of reference is a rigid scaffolding, the prototype of a Relativistic one is the radar station. Secondly, the class of transformations permissible in Newtonian mechanics is much wider than in the Theory of Relativity; Newtonian mechanics does not have to consider the limitations, discussed above, which rise from the existence of a limiting speed.

As an example we consider a transformation which can be interpreted in Newtonian mechanics as going over to a uniformly accelerated frame. Let x', y', z' and t' be the space and time coordinates in an inertial frame, i.e. Galilean coordinates. We put

$$x' = x - \frac{1}{2}at^2; \quad y' = y; \quad z' = z \qquad (35.42)$$

and also

$$t' = t - \frac{a}{c^2}tx \qquad (35.43)$$

The variables x, y, z and t can be interpreted as space and time coordinates in a certain accelerated frame (in the Newtonian sense and in the corresponding approximation). Inserting (35.42) and (35.43) into the expression for ds^2 we get

$$ds^2 = (c^2 - 2ax - a^2t^2)\,dt^2 - dx^2 - dy^2 - dz^2 + \frac{a^2}{c^2}(x\,dt + t\,dx)^2 \qquad (35.44)$$

The required inequalities for the coefficients will hold if the conditions

$$1 - \frac{a^2t^2}{c^2} > 0; \quad \left(1 - \frac{ax}{c^2}\right)^2 - \frac{a^2t^2}{c^2} > 0 \qquad (35.45)$$

are satisfied. In addition we can require that

$$\frac{\delta t'}{\delta t} = 1 - \frac{ax}{c^2} > 0 \qquad (35.46)$$

These inequalities show that the substitutions (35.42), (35.43) are permissible only in a part of space and only for a limited length of time.

Another example is the transformation corresponding to the introduction of a uniformly rotating frame. We put

$$x' = x \cos \omega t + y \sin \omega t; \quad z' = z \qquad (35.47)$$
$$y' = - x \sin \omega t + y \cos \omega t; \quad t' = t$$

and obtain

$$ds^2 = [c^2 - \omega^2(x^2 + y^2)]\, dt^2 - 2\, \omega\, (ydx - xdy)\, dt - dx^2 - dy^2 - dz^2 \qquad (35.48)$$

The conditions on the coefficients require

$$c^2 - \omega^2(x^2 + y^2) > 0 \qquad (35.49)$$

which is satisfied only for distances from the axis of rotation less than that where the linear velocity of the rotation equals the speed of light.

We stress once again that the examples given here have physical sense only in a region in which Newtonian mechanics is applicable [(see also Section 61).]

It is obvious that the introduction of ordinary curvilinear spatial coordinates is always an allowed transformation. As long as the transformations do not involve time they have the same geometrical meaning as in non-relativistic theory. Therefore we refrain from discussing them.

36. General Tensor Analysis and Generalized Geometry

In the previous section we considered the expressions

$$(\nabla \omega)^2 = \sum_{\alpha\beta=0}^{3} g^{\alpha\beta} \left(\frac{\delta \omega}{\delta x_\alpha} \frac{\delta \omega}{\delta x_\beta} \right) \qquad (36.01)$$

and

$$ds^2 = \sum_{\alpha\beta=0} g_{\alpha\beta} dx_\alpha dx_\beta \qquad (36.02)$$

which were obtained from the usual expressions of Relativity Theory by introducing variables x1, x2, x3, and x0 in place of the space and time coordinates x, y, z and t. We established the conditions subject to which the variable x_0 can characterize a sequence of events in time and the variables x_1, x_2, and x_3 their location in space.

By itself, the introduction of new variables can naturally not influence the physical consequences of the theory; it is merely a mathematical device. However, the development of a formalism which permits the statement of equations of mathematical physics (such as equations of motion and field equations) directly in terms of arbitrary variables without the use of Cartesian spatial coordinates and time, is not only useful as a device for convenient computation but is also important in principle. The existence of such a formalism can show the way to generalize physical theories.

We shall call equations generally covariant, if they are valid for any arbitrary choice of independent variables. The formalism that allows anyone to write down generally covariant tensor equations will be called "general tensor analysis."

Generally covariant equations are already used in Newtonian mechanics. We refer to Lagrange's equations (of the second kind) which describe the motion of a system of mass points in generalized coordinates and also their generalization for continuous media. While they state nothing physically new as compared to equations in Cartesian coordinates, Lagrange's equations nevertheless play an important part both in practical applications and in theoretical investigations. In the Theory of Relativity general tensor analysis has a similar purpose.

In general tensor analysis the starting point is the pair of equations (36.01) and (36.02) giving the square of the four-dimensional gradient and the square of the interval. One says that these expressions characterize the *metric* of spacetime. The coefficients $g^{x\beta}$ and $g_{\alpha\beta}$ entering the equations are thought of as functions of the variables x0, x1, x2, and x3.

We have so far assumed that the expressions (36.01) and (36.02) are derived from (35.01) and (35.02), or from (35.08) and (35.09), by introduction of new variables, so that the coefficients $g^{x\beta}$ and $g_{\alpha\beta}$ can be represented in the terms of the four functions f0, f1, f2, and f3 as follows:

$$g_{\alpha\beta} = \sum_{k=0}^{3} ek \left(\frac{\delta f_k}{\delta f_k} \frac{\delta x f}{\delta x_\beta} \right) \qquad (36.03)$$

By virtue of (35.16) the $g^{x\beta}$ can be expressed in terms of the same four functions.

However, it is important to note that the equations of general tensor analysis are hardly any more complicated if it is not assumed that the $g_{\alpha\beta}$ can be represented in the form (36.03) but that instead they are taken simply as given functions of the coordinates, i.e. of the variables, x0, x1, x2, and x3. This more general point of view corresponds to the introduction of non-Euclidean geometry and a non-Euclidean metric in spacetime. Such a step takes one beyond the limits of ordinary (so-called "Special") Relativity and is connected with the formulation of a new physical theory, namely Einstein's Theory of Gravitation [which is described in Appendix VI]. [Later chapters of this book are devoted to this theory, but in this chapter,] presently we adopt a purely formal view and develop general tensor analysis on the assumption that the metric is given and the $g_{x\beta}$ are known functions of the coordinates. Such a presentation has two advantages. In the first place we can find the conditions which the $g_{\alpha\beta}$ must satisfy in order to be expressible in the form (36.03); this gives us a generally covariant formulation of the usual Theory of Relativity. In the second place we obtain in this way the mathematical apparatus for formulating Einstein's Theory of Gravitation.

Before going on to a systematic exposition of general tensor analysis we establish the connection between the expressions $(\nabla\omega)^2$ and ds^2 which exists if

$$\sum_{\alpha=0}^{3} g_{v\alpha} g^{v\alpha} = \delta_{\mu}^{v} \qquad (36.04)$$

regardless of whether the $g_{\alpha\beta}$ are of the form (36.03) or not. We show that if a function $\omega(x0, x1, x2, x3)$ satisfies $(\nabla\omega)^2 = 0$ then the differentials of the coordinates related by the condition ω = const. satisfy $ds^2 = 0$.

Putting

$$\omega_\alpha = \frac{d\omega}{dx_\alpha} \quad (\alpha = 0,1,2,3) \qquad (36.05)$$

We write $(\nabla\omega)^2 = 0$ in the form

$$G \equiv \sum_{\alpha\beta=0}^{3} g^{\alpha\beta}\omega_\alpha\omega_\beta = 0 \qquad (36.06)$$

This partial differential equation for ω is of the same type as the Hamilton-Jacobi equation of classical mechanics and can be solved similarly to the latter. If we solve it for ω_0 and write

$$\omega_0 = - H(\omega_1, \omega_2, \omega_3) \qquad (36.07)$$

the function H will correspond to the Hamiltonian and Hamilton's equations will be

$$\frac{dx_k}{dx_0} = \frac{\delta H}{\delta\omega_k}; \quad \frac{d\omega_k}{dx_0} = -\frac{\delta H}{\delta x_k} \quad (k = 1, 2, 3) \qquad (36.08)$$

But

$$\frac{\delta H}{\delta\omega_k} = -\frac{d\omega_0}{d\omega_k} = \frac{\delta G/\delta\omega_k}{\delta G/\delta\omega_0} \qquad (36.09)$$

and the first three equations of (36.05) show that the differentials $dx_\alpha(\alpha = 0, 1, 2, 3)$ are proportional to the partial derivatives of G with respect to the ω_α. Denoting the infinitesimal coefficient of proportionality by *dp, we have

$$dx_\alpha = \frac{dp}{2}\frac{\delta G}{\delta\omega_\alpha} = dp\sum_{\beta=0}^{3} g^{\alpha\beta}d\omega_\beta \qquad (36.10)$$

Solving for ω_α with the use of (36.04) we get

$$\omega_\alpha dp = \sum_{\beta=0}^{3} g_{\alpha\beta}dx_\beta \qquad (36.11)$$

and the obvious relation

$$\sum_{\alpha=0}^{3} \omega_\alpha dx_\alpha = 0 \qquad (36.12)$$

then gives

$$ds^2 = \sum_{\alpha\beta=0}^{3} g_{\alpha\beta}dx_\alpha dx_\beta = 0 \qquad (36.13)$$

as required. Thus if we continue to consider the equation $(\nabla\omega)^2 = 0$ as describing a wavefront we can take it that for points on the wavefront the differentials of space and time coordinates are related by $ds^2 = 0$.

In the following we shall consider the $g_{x\beta}$ as given functions of the variables x_0, x_1, x_2, and x_3 and shall merely assume that they have continuous derivatives of all orders considered and that they satisfy the inequalities stated in Section 35. In addition to the $g_{\alpha\beta}$ we shall consider the $g^{\alpha\beta}$, their connection being (36.04). The conditions under which the $g_{\alpha\beta}$ can be represented in the form (36.03) [will be] <u>is</u> established in Section 42 <u>of the following reference which is herein incorporated by reference in its entirety: The Theory of Space, Time and Gravitation, V. Fock, the MacMillan Company, New York, 1964. In particular, Section 20 of this reference gives the definition of a four-dimensional vector; Section 21 describes four-dimensional tensors; Section 22 describes pseudo-tensors; Section 37 describes the definition of a vector and of a tensor in terms of tensor algebra; Section 42 describes the transformation law for Christoffel symbol, and the locally geodesic coordinate system, conditions for transforming ds^2 to a form with constant coefficients; Section 43 describes the curvature tensor; Section 44 describes the properties of the curvature tensor, and Section 31 describes the mass tensor. Also Section 39 describes the parallel transport of a vector; and Section 40 describes covariant differentiation.</u>

38. The Equation of a Geodesic

We consider two point-instants corresponding to two events in time sequence and we denote their coordinates by $x_\alpha^{(1)}$ and $x_\alpha^{(2)}$ respectively. Let a material point move along some curve in such a way that when $x_0 = x_0^{(1)}$ its spatial coordinates are $x_k = x_k^{(1)}$ and when $x_0 = x_0^{(2)}$ they are $x_k = x_k^{(2)}$.

As the events $x_\alpha^{(1)}$ and $x_\alpha^{(2)}$ are assumed to be in time sequence such motion is possible with a speed less than that of light. The time x_0 and the spatial coordinates x_k corresponding to it can be expressed parametrically in terms of an auxiliary variable p, by putting

$$x_\alpha = \varphi^\alpha (p) \qquad (38.01)$$

with

$$\overset{(1)}{\underset{\alpha}{x}} = \varphi(p_1); \quad \overset{(2)}{\underset{\alpha}{x}} = \varphi^\alpha(p_2) \qquad (38.02)$$

Since the speed of the motion is less than that of light the inequality

$$ds^2 = g_{\alpha\beta}\, \dot\varphi^\alpha \dot\varphi^\beta\, dp^2 > 0 \qquad (38.03)$$

must hold for any infinitesimal interval along the path. Here a dot denotes differentiation with respect to p. The finite interval between the events in time sequence which is proportional to the interval of proper time τ, will be denoted by $s = c\tau$ and we have

$$s = c\tau = \int_{p_1}^{p_2} = \sqrt{(g_{\alpha\beta}\, \dot\varphi^\alpha \dot\varphi^\beta)} \cdot dp \qquad (38.04)$$

We now consider two quasi-simultaneous events. The two points in space at which the events take place can be joined by some curve and to each joint on this curve we can assign a definite instant of time, i.e. we can write down the "time equation" for each point, taking care that any two intermediate space-instants are quasi-simultaneous. The analytic expressions for the curve and the time equation may again be stated in the form of equations (38.01) and (38.02), but we can no longer interpret these equations as describing the *motion* of a point along a curve; they now give a *static* description of the curve as a whole. For any pair of intermediate points, infinitesimally separated, we have

$$ds^2 = g_{\alpha\beta}\, \dot{\varphi}^\alpha\, \dot{\varphi}^\beta\, dp^2 < 0 \qquad (38.05)$$

and the space-like interval

$$l = \int_{P1}^{P2} = \sqrt{(-g_{\alpha\beta}\, \dot{\varphi}^\alpha\, \dot{\varphi}^\beta)} \cdot dp \qquad (38.06)$$

characterizes the length of the curve.

The question arises of the extremal values of both the time-like interval (38.04) between two events in time sequence and the space-like interval (38.06) between two quasi-simultaneous events. Both these variational problems lead to equations of the same form, whether the interval is time-like or space-like. The variational equations are called the equations of the geodesic by analogy with the theory of surfaces. However, it is important to note that whereas in the theory of surfaces, where the square of an infinitesimal distance is a positive definite quadratic form of the coordinate differentials, the geodesic is, generally speaking*, a *shortest* line; in the four-dimensional spacetime manifold the situation is different; the extremal value of the interval is a *maximum* for a time-like interval and neither a maximum nor a minimum for a space-like interval. This can easily be verified in the special case of the Galilean metric where ds^2 has the form (37.04). For events in time sequence we can then choose a reference frame so that the spatial coordinates of the initial and final points are the same and we can choose the time t as the parameter. We then have

$$s = \int_{t^{(1)}}^{t^{(2)}} \sqrt{(c^2 - v^2)} \cdot dt \qquad (38.07)$$

where

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \qquad (38.08)$$

The solution of the variational problem in this case is given by constant values of x, y and z, so that $v^2 = 0$. For any other trajectory v^2 will somewhere be greater than zero, so that $\sqrt{c^2 - v^2}$, c and therefore

$$s < s_{max} = c(t^{(2)} - t^{(1)}) \qquad (38.09)$$

For the space-like interval we can choose a frame of reference such that

$$t^{(2)} = t^{(1)}; \quad y^{(2)} = y^{(1)}; \quad z^{(2)} = z^{(1)} \qquad (38.10)$$

while $x^{(2)} > x^{(1)}$. Taking the coordinate x as the parameter we obtain

---

* i.e. for sufficiently near terminal points.

$$l = \int_{x^{(1)}}^{x^{(2)}} \sqrt{(1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dt}{dx}\right)^2)} \cdot dx \qquad (38.11)$$

The solution of the variational problem is now given by constant values of y, z and t for which then

$$l_{extr} = x^{(2)} - x^{(1)} \qquad (38.12)$$

However, the other curves y(x), z(x) or other time equations t(x) we may find either $l > l_{extr}$ or $l < l_{extr}$ depending on whether the square root in (38.11) is in the mean greater or less than unity.

We now derive the differential equations of the geodesic. The Lagrangian of the variational problem is

$$L = \sqrt{(\dot{g}_{\alpha\beta}\, \dot{\varphi}^{\alpha}\, \dot{\varphi}^{\beta})} \qquad (38.13)$$

or, writing x_α instead of ϕ^α

$$L = \sqrt{(g_{\alpha\beta}\, \dot{x}_\alpha\, \dot{x}_\beta)} \qquad (38.14)$$

The extremal condition for the integral

$$s = \int_{P1}^{P2} L\, dp \qquad (38.15)$$

leads to the Euler-Lagrange equations

$$\frac{d}{dp}\frac{\delta L}{\delta \dot{x}_\alpha} - \frac{\delta L}{\delta x_\alpha} = 0 \qquad (38.16)$$

We now put

$$F = \frac{1}{2} g_{\alpha\beta}\, \dot{x}_\alpha\, \dot{x}_\beta \qquad (38.17)$$

so that $L = \sqrt{(2F)}$ (38.18)

[By the same reasoning as in Section 17 we] we can choose a parameter p so that

$$\frac{dF}{dp} = 0; \quad F = \text{const} \qquad (38.19)$$

and with this choice (38.16) is equivalent to

$$\frac{d}{dp}\frac{\delta F}{\delta \dot{x}_\alpha} - \frac{\delta F}{\delta x_\alpha} = 0 \qquad (38.20)$$

These last equations possess the integral

$$\dot{x}_\alpha \frac{\delta F}{\delta \dot{x}_\alpha} - F = F = \text{const} \qquad (38.21)$$

so that the condition (38.19) is a consequence of (38.21). Inserting the explicit expression for F we obtain from (38.20)

$$\frac{d}{dp}(g_{\alpha\beta}\, \dot{x}_\beta) - \frac{1}{2}\frac{\delta g_{\beta\gamma}}{dx_\alpha}\dot{x}_\beta\, \dot{x}_\gamma = 0 \qquad (38.22)$$

or, performing the differentiations,

$$g_{\alpha\beta}\ddot{x}_\beta + \frac{\delta g_{\alpha\beta}}{dx_\gamma}\dot{x}_\beta\dot{x}_\gamma - \frac{1}{2}\frac{\delta g_{\beta\gamma}}{dx_\alpha}\dot{x}_\beta\dot{x}_\gamma = 0 \qquad (38.23)$$

The coefficient of $\ddot{x}_\beta\dot{x}_\gamma$ can be symmetrized with respect to β and γ; if we put

$$[\beta\gamma, \alpha] = \frac{1}{2}\left(\frac{\delta g_{\alpha\beta}}{\delta x_\gamma} + \frac{\delta g_{\alpha\gamma}}{\delta x_\beta} - \frac{\delta g_{\beta\gamma}}{\delta x_\alpha}\right) \qquad (38.24)$$

the differential equation of the geodesic becomes

$$g_{\alpha\beta}\ddot{x}_\beta + [\beta\gamma, \alpha]\ \dot{x}_\beta\dot{x}_\gamma = 0 \qquad (38.25)$$

The expression (38.24) is called a Christoffel symbol of the first kind. In order to solve equations (38.25) for the second derivatives we multiply them by $g^{\alpha\nu}$ and sum over α. Then with the new symbol

$$\{\beta\gamma, \nu\} = g^{\alpha\nu}[\beta\gamma, \alpha] \qquad (38.26)$$

we obtain

$$\ddot{x}_\nu + \{\beta\gamma, \nu\}\dot{x}_\beta\dot{x}_\gamma = 0 \qquad (38.27)$$

The expression (38.26) is called a Christoffel symbol of the second kind and is often represented by an alternative symbol

$$\{\beta\gamma, \nu\} = \Gamma^{\nu}_{\beta\gamma} \qquad (38.28)$$

For uniformity we can also introduce a corresponding form of notation for Christoffel symbols of the first kind:

$$[\alpha\beta, \gamma] = \Gamma_{\gamma,\alpha\beta} \qquad (38.29)$$

but this is less common practice.

We have thus

$$\Gamma_{\nu,\alpha\beta} = \frac{1}{2}\left(\frac{\delta g_{\nu\alpha}}{\delta x_\beta} + \frac{\delta g_{\nu\beta}}{\delta x_\alpha} - \frac{\delta g_{\alpha\beta}}{\delta x_\nu}\right) \qquad (38.30)$$

and

$$\Gamma^{\nu}_{\beta\gamma} = \frac{1}{2}g^{\nu\mu}\left(\frac{\delta g_{\mu\alpha}}{\delta x_\beta} + \frac{\delta g_{\mu\beta}}{\delta x_\alpha} - \frac{\delta g_{\alpha\beta}}{\delta x_\mu}\right) \qquad (38.31)$$

In this notation the equation of a geodesic takes on the form

$$\frac{d^2x_\nu}{dp^2} + \Gamma^{\nu}_{\alpha\beta}\frac{dx_\alpha}{dp}\frac{dx_\beta}{dp} = 0 \qquad (38.32)$$

If the Christoffel symbols correspond to a metric tensor that can be written in the form (35.15) equations (38.32) are equivalent to the relations

$$\frac{d^2x'_\kappa}{dp^2} = 0 \qquad (k = 0, 1, 2, 3) \qquad (38.33)$$

for the Galilean coordinates x'_k. This follows from the covariance of the equations and the fact that in Galilean coordinates the Christoffel symbols vanish. In this case, therefore, the equation of a geodesic leads to linear dependence of the Galilean coordinates on the parameter p.

It is not difficult to verify that the development leading to (38.32) remains valid whatever the sign of F. If $F > 0$ the "geodesic" joins two events in time sequence and equations (38.32) can be interpreted as the equation of motion of a free mass point moving with a speed less than that of light. The increment dp of p is proportional to the increment $d\tau$ of the proper time τ and (38.32) may be replaced by

$$\frac{d^2x_\nu}{d\tau^2} + \Gamma^\nu_{\beta\gamma}\frac{dx_\alpha}{d\tau}\frac{dx_\beta}{d\tau} = 0 \qquad (38.34)$$

The length of the geodesic gives the interval of proper time between the "departure" and the "arrival" of the mass point. If on the other hand $F < 0$, the geodesic joins two quasi-simultaneous events and we can put dp equal to the increment of the spatial interval. Equations (38.32) then appear as

$$\frac{d^2x_\nu}{dl^2} + \Gamma^\nu_{\alpha\beta}\frac{dx_\alpha}{dl}\frac{dx_\beta}{dl} = 0 \qquad (38.35)$$

The case $F = 0$ corresponds to a point moving along a ray with the speed of light. In this case the Lagrangian (38.18) is zero and the above derivation of the geodesic equation is no longer valid. However, the equations (38.32) themselves retain their meaning and as they possess the integral (38.21) they are compatible with the condition $F = 0$. To justify the equations in this case we can start from the Hamiltonian equations that were discussed in Section 36. According to (36.08) we have

$$\frac{dx_k}{dx_0} = \frac{\delta H}{\delta\omega_k}, \quad \frac{\delta\omega_k}{\delta x_0} = -\frac{\delta H}{\delta x_k} \qquad (k = 1,2,3) \qquad (38.36)$$

where the Hamiltonian $H = -\omega_0$ is obtained by solving for ω_0 the equation

$$G \equiv g^{\alpha\beta}\omega_\alpha\omega_\beta = 0 \qquad (38.37)$$

Therefore we have

$$dH = -d\omega_0 = \frac{1}{\delta G/\delta\omega_0}\left(\frac{\delta G}{\delta x_\alpha}\delta x_\alpha + \frac{\delta G}{\delta\omega_k}\delta\omega_k\right) \qquad (38.38)$$

Using the fact that

$$\frac{d\omega_0}{dx_0} = -\frac{dH}{dx_0} = -\frac{\delta H}{\delta x_0} \qquad (38.39)$$

and expressing the derivatives of H in terms of the derivatives of G we can write equations (38.36) in a symmetric fashion:

$$\frac{dx_\alpha}{dp} = \frac{1}{2}\frac{\delta G}{\delta\omega_\alpha}, \quad \frac{d\omega_\alpha}{dp} = -\frac{1}{2}\frac{\delta G}{\delta x_\alpha} \qquad (\alpha = 0, 1, 2, 3) \qquad (38.40)$$

Here dp is considered to be the differential of the independent variable p. The first four equations of (38.40) have already been given in Section 36. Writing the right-hand sides explicitly we get

$$\frac{dx_\alpha}{dp} = g^{\alpha\beta}\omega_\beta; \quad \frac{d\omega_\alpha}{dp} = -\frac{1}{2}\frac{\delta g^{\mu\nu}}{\delta x_\alpha}\omega_\mu\omega_\nu \qquad (38.41)$$

It is readily seen that these equations are equivalent to (38.32), for we have

$$\omega_\mu = g_{\mu\lambda}\frac{dx_\lambda}{dp} \qquad (38.42)$$

and therefore

$$\frac{\delta g^{\mu\nu}}{\delta x_\alpha}\omega_\mu = \frac{\delta g^{\mu\nu}}{\delta x_\alpha} g_{\mu\lambda}\frac{dx_\lambda}{dp} = -g^{\mu\nu}\frac{dg_{\mu\lambda}}{dx_\alpha}\frac{dx_\lambda}{dp} \qquad (38.43)$$

since

$$\frac{\delta g^{\mu\nu}}{\delta x_\alpha} g_{\mu\lambda} + g^{\mu\nu}\frac{\delta g_{\mu\lambda}}{\delta x_\alpha} = \frac{\delta}{\delta x_\alpha}(g^{\mu\nu} g_{\mu\lambda}) = \frac{\delta}{\delta x_\alpha}(\delta^\nu{}_\lambda) = 0 \qquad (38.44)$$

Inserting (38.43) into (38.41) we get

$$\frac{d\omega_\alpha}{dp} = \frac{1}{2} g^{\mu\nu}\omega_\nu\frac{\delta g_{\mu\lambda}}{\delta x_\alpha}\frac{dx_\lambda}{dp} \qquad (38.45)$$

or, in consequence of the first set of equations in (38.41),

$$\frac{d\omega_\alpha}{dp} = \frac{1}{2}\frac{\delta g_{\mu\lambda}}{\delta x_\alpha}\frac{dx_\mu}{dp}\frac{dx_\lambda}{dp} \qquad (38.46)$$

Eliminating the ω_α from these equations and (38.42) we finally obtain

$$\frac{d}{dp}\left(g_{\alpha\beta}\frac{dx_\beta}{dp}\right) = \frac{1}{2}\frac{\delta g_{\mu\lambda}}{\delta x_\alpha}\frac{dx_\mu}{dp}\frac{dx_\lambda}{dp} \qquad (38.47)$$

These equations are the same as the equations (38.22) from which the equations of the geodesic in the form (38.32) were derived. The passage from (38.41) to (38.47) is the usual one from Hamilton's to Lagrange's equations.

We have thus proved that the geodesic of zero length is likewise determined by the equations (38.22) but with the condition F = 0 adjoined.

It should be noted that because F is constant, a geodesic retains its character for its entire length; it may always describe the motion of a point with a speed less than that of light or it may be a null-line or, finally, it may be everywhere space-like.

For a null-geodesic the relation (38.37) with $\omega_\alpha = \frac{\delta\omega_\alpha}{\delta x_\alpha}$ may be considered to be the Hamilton-Jacobi equation for the action function ω. (See Section 36.) The Hamilton-Jacobi equation for the general case can also be readily obtained. For definiteness we consider the case of a point moving with a speed less than that of light.

Choosing the time $t = x_0$ as the parameter and denoting differentiation with respect to it by a dot we can write the Lagrangian of the problem* in the form

* It is convenient to introduce the Lagrangian with opposite sign to the equal convention in mechanics. As a result, the sign of the energy will be opposite to the sign of the Hamiltonian.

$$L = +\sqrt{(g_{00} + 2g_{0i}\dot{x}_i + g_{ik}\dot{x}_i\,\dot{x}_k)} \qquad (38.48)$$

The generalized momenta are

$$\frac{\delta L}{\delta \dot{x}_i} = p_i = +\frac{1}{L}(g_{0i} + g_{ik}\,\dot{x}_k) \qquad (38.49)$$

and the Hamiltonian is found by the usual rule to be the expression

$$H\,\dot{x}_i p_i - L = -\frac{1}{L}(g_{oo} + g_{ok}\,\dot{x}_k) \qquad (38.50)$$

with the velocities x_k expressed in terms of the momenta p_t by (38.49).

If we put

$$p_0 = \frac{1}{L}(g_{00} + g_{0k}\,\dot{x}_k) \qquad (38.51)$$

and observe that

$$Ldt = ds \qquad (38.52)$$

where s is the length of arc, the four quantities $p_t p_0$ can be uniformly written as

$$p_\alpha = g_{\alpha\beta}\frac{dx_\beta}{ds} \qquad (38.53)$$

The identity

$$g_{\alpha\beta}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds} = 1 \qquad (38.54)$$

leads to the relation

$$g^{\mu\nu}p_\mu p_\nu = 1 \qquad (38.55)$$

which can be regarded as the result of eliminating the three velocities $\dot{x}_1$, $\dot{x}_2$, and $\dot{x}_3$ from the four equations (38.49) and (38.51). The Hamiltonian $H = -p_0$ is obtained by solving (38.55) for p_0. The Hamilton-Jacobi equation is obtained by the usual rule of expressing p_1, p_2, p_3 and H as partial derivatives of S with respect to the spatial coordinates and to time, as follows:

$$H = -\frac{\delta S}{\delta t}, \quad p_k = \frac{\delta S}{\delta x_k} \qquad (38.56)$$

These equations can also be written as

$$p_\nu = \frac{\delta S}{\delta x_\nu} \qquad (38.57)$$

Thus the Hamilton-Jacobi form of the equation of a geodesic is

$$g_{\mu\nu} = \frac{\delta S}{\delta x_\mu}\frac{\delta S}{\delta x_\nu} = 1 \qquad (38.58)$$

If a complete integral of the Hamilton-Jacobi equation

$$S = S(x_0, x_1, x_2, x_3, c_1, c_2, c_3) + c_0 \qquad (38.59)$$

is known, which contains three arbitrary constants c_1, c_2, and c_3, not counting the additive constant c_0, the derivatives of S with respect to the constants,

$$\frac{\delta S}{\delta c_k} = b_k \qquad (k = 1, 2, 3) \qquad (38.60)$$

are also constants, as is proved in mechanics. They are determined from the conditions of the problem.

A comparison of (38.58) with (38.37) shows that the equations of a null-geodesic are obtained from (38.58) by replacing the right-hand side by zero. For a space-like geodesic the right-hand side of the Hamilton-Jacobi equation is a negative constant which can be set equal to -1.

APPENDIX V

(The following text is published in The Theory of Space, Time and Gravitation, Chapter IV, pp. 178-182 and is incorporated herein by reference.

According to the present invention, departures from the text have been made. Additions are underlined and deletions are [bracketed] and initialed.)

[.] Remark on the Relativity Principle and the Covariance of equations

[At the beginning of the book (Section 6) we gave] In Appendix I a formulation of the principle of relativity, was given which together with the postulate that the velocity of light has a limiting character, may be made the basis of relativity theory. We shall now investigate in more detail the question of the connection of the physical principle of relativity with the requirement that the equations be covariant. [We have already touched on this problem in the Introduction.]

In the first place, we shall attempt to give a generally covariant formulation of the principle of relativity, without as yet making this concept more precise. In its most general form the principle of relativity states the equivalence of the coordinate systems (or frames of reference) that belong to a certain class and are related by transformations of the form

$$x'_\alpha = f_\alpha(x_0, x_1, x_3) \qquad (49^*.01)$$

which may be stated more briefly as

$$x' = f(x) \qquad (49^*.02)$$

It is essential to remember that, in addition to the group of permissible transformations, the class of coordinate systems must be characterized by certain supplementary conditions. Thus, for instance, if we consider Lorentz transformations, it is self-evident that these linear transformations must connect not any arbitrary coordinates, but only the Galilean coordinates in two inertial reference frames. To consider linear transformations between any other (non-Galilean) coordinates has no sense, because the Galilean principle of relativity has no validity in relation to such artificial linear transformations. On the other hand, if one introduces any other variables in place of the Galilean coordinates, a Lorentz transformation can evidently be expressed in terms of these variables, but then the transformation formulae will have a more complicated form.

Let us now state more precisely what is meant in the formulation of the principle of relativity by equivalence of reference frames. Two reference frames (x) and (x') may be called physically equivalent if phenomena proceed in the same way in them. This means that if a possible process is described in the coordinates (x) by the functions

$$\varphi_1 (x), \varphi_2(x),....,\varphi_n(x) \qquad (49^*.03)$$

then there is another possible process which is describable by the same functions

$$\varphi_1 (x'), \varphi_2(x'),.....,\varphi_n(x') \qquad (49^*.04)$$

in the coordinates (x'). Conversely any process of the form (49.04) in the second system corresponds to a possible process of the form (49*.03) in the first system. [Such a definition of corresponding processes agrees fully with that given in Section 6.] Thus a relativity principle is a statement concerning the existence of corresponding processes in a set of reference frames of a certain class; the systems of this class are then accepted as equivalent. It is clear from this definition that both the principle of relativity itself and the equivalence of two reference frames are physical concepts, and the statement that the one or the other is valid involves a definite physical hypothesis and is not just conventional. In addition, it follows that the very notion of a "principle of relativity" becomes well-defined only when a definite class of frames of reference has been singled out. In the usual theory of relativity this class is that of inertial systems.

The functions (49*.03) or (49*.04) describing a physical process will be called field functions or functions of state. [We have already indicated in Section 46 that in] <u>In</u> a generally covariant formulation of the equations describing physical processes, the components $g_{\mu\nu}$ of the metric tensor must be included among the functions of state. [In the example there discussed we] <u>We</u> then get the following collection of field functions:

$$F_{\mu\nu}(x), j_\nu(x), g_{\mu\nu}(x) \qquad (49^*.05)$$

[i.e.] <u>which are</u> the electromagnetic field, the current vector and the metric tensor. The requirement entering the formulation of a principle of relativity that in the two equivalent reference frames corresponding phenomena should proceed in the same way applies equally to the metric tensor. Thus if we compare two corresponding phenomena in two physically equivalent reference frames, then for the first phenomenon, described in the old coordinates, not only the components of electromagnetic field and of current density, but also the components of the metric tensor must have the same mathematical form as for the second phenomenon described in the new coordinates.

What can be concluded further will depend on whether we assume that the metric is fixed or whether we take into consideration phenomena which themselves influence the metric. In the [usual] <u>special</u> theory of relativity [described in the previous chapters] it is assumed that the metric is given once and for all and does not depend on any physical processes. The generally covariant formulation of the theory of relativity given [in the present chapter] <u>presently</u> does not change anything in this. As long as the assumption remains in force that the character of spacetime is Galilean and the $g_{\mu\nu}$ are introduced only to achieve general covariance, these quantities will depend only on the choice of coordinate system, not on the nature of the physical process discussed; they are functions of state only in a formal sense. In the theory of gravitation on the other hand, to the description of which we turn in the following [chapter] <u>Appendices VI and VII</u>, a different assumption is made concerning the nature of spacetime. There the $g_{\mu\nu}$ are the functions of state, not only in the formal sense, but in fact: they describe a certain physical field, namely the field of gravitation. However, when discussing small-scale processes which do not influence the motion of heavy masses one can also assume that the metric is fixed (though not Galilean).

To give a definite meaning to the principle of relativity in such circumstances, it is essential to specify more closely not only the class of coordinate systems, but also the nature of the physical processes from which the principle is being formulated.

We shall first start from the assumption that the metric is fixed ("rigid"), or else that it may be considered as fixed for a certain class of physical processes. We return to the above definition of corresponding phenomena in two physically equivalent coordinate systems, according to which all field functions, including the components of the metric tensor, must have the same mathematical form for the first process described in the old coordinates os for the second process described in the new coordinates. If the $g_{\mu\nu}$ are independent of the nature of the physical phenomenon, then in relation to these quantities we need not make a distinction between the first and the second process, and need consider only transformations of the coordinates. But the quantities

$$g_{\mu\nu}(x) \text{ and } g'_{\mu\nu}(x') \qquad (49^*.06)$$

will be connected by the tensor transformation rule; the requirement of the relativity principle that they should have one and the same mathematical form reduces (for infinitesimal coordinate transformations) to the equations $\delta g_{\mu\nu} = 0$. [discussed in Sections 48 and 49.]

We know that the most general class of transformations that satisfies these equations contains 10 parameters and is possible only in uniform spacetime, where the relation

$$R_{\mu\nu,\alpha\beta} = K \cdot (g_{\nu\alpha}g_{\mu\beta} - g_{\mu\alpha}g_{\nu\beta}) \qquad (49^*.07)$$

[(see equation 49.12)] is valid. If in these relations K is zero, the spacetime is Galilean and the transformations in question are Lorentz transformations, except that possibly they may be written down in other (non-Galilean) coordinates.

Thus with the rigidity assumption for the metric, the principle of relativity implies the uniformity of spacetime, and if the additional condition K = 0 holds, we obtain a Galilean metric in appropriate coordinates. The relativity principle in general form then reduces to the Galilean relativity principle. As for the condition K = 0, it results in an additional uniformity of spacetime; if the scale of the Galilean coordinates is changed, than the scale of the elementary interval changes in the same proportion. [ ] This property implies in turn that there is no absolute scale for spacetime, unlike the absolute scale that exists for velocities in terms of the velocity of light; the absence of an absolute scale for spacetime leads conversely to the equation K = 0.

If we now go over to discuss phenomena which may themselves influence the metric, we must reckon with the possibility that under certain conditions the principle of relativity will be valid in non-uniform space also. For this to be so, it is necessary that the motion of the masses producing the non-uniformity be included in the description of the phenomena.

Indeed, [at the end of this book it will] it can be shown that under the assumption that spacetime is uniform at infinity (where it must be Galilean) one can single out a class of coordinate systems that are analogous to inertial systems and defined up to a Lorentz transformation. In relation to this class of coordinate systems a principle of relativity will hold in the same form as in the usual theory of relativity, in spite of the fact that at a finite distance from the masses the space if non-uniform. Here however one must bear in mind the essential role played by the boundary conditions that require uniformity at infinity. Thus in the last analysis the relativity principle is here also a result of uniformity.

Since the greatest possible uniformity is expressed by Lorentz transformations there cannot be a more general principle of relativity than that discussed in ordinary relativity theory. All the more, there cannot be a general principle of relativity, as a physical principle, which would hold with respect to arbitrary frames of reference.

In order to make this fact clear, it is essential to distinguish sharply between a physical principle that postulates the existence of corresponding phenomena in different frames of reference and the simple requirement that equations should be covariant in the passage from one frame of reference to another. It is clear that a principle of relativity implies a covariance of differential equations, but the converse is not true: covariance of differential equations is possible also when no principle of relativity is satisfied. Quite apart from the fact that not all laws of nature reduce to differential equations, even fields described by differential equations require for their definitions not only these equations, but also all kinds of initial, boundary and other conditions. These conditions are not covariant. Therefore the preservation of their physical content requires a change in their mathematical form and, conversely, preservation of their mathematical form implies a change of their physical content. But the realizability of a process with a new physical content is an independent question which cannot be solved a priori. If within a given class of reference "corresponding" physical processes are possible, then a principle of relativity holds. In the opposite case it does not. It is clear, however, that such a model representation of physical processes, and in particular such a model representation of the metric, is possible at most for a narrow class of reference systems, and certainly cannot be unlimited. This argument shows once again (without invoking the concept of uniformity) that a general principle of relativity, as a physical principle, holding in relation to arbitrary frames of reference, is impossible.

[It should be remembered that in the general case the expression for ds^2, though always a homogeneous function of the coordinate differentials, may also depend in a non-homogeneous way on the coordinates themselves.]

But as a motivation of the requirement of covariance of the equations a general principle of relativity is also unnecessary. The covariance requirement can be justified independently. It is a self-evident, purely logical requirement that in all cases in which the coordinate system is not fixed in advance, equations written down in a different coordinate system should be mathematically equivalent. The class of transformations with respect to which the equations must be covariant must correspond to the class of coordinate systems considered. Thus if one deals with inertial systems related by Lorentz transformations and if Galilean coordinates are used, it is sufficient to require covariance with respect to Lorentz transformations.

[(as was done in Chapters I and II of this book)] If, however, arbitrary coordinates are employed, it is necessary to demand general covariance [(Chapter IV)].

It should be noted that covariance of coordinate systems acquires a definite physical meaning if, and only if, a principle of relativity exists for the class of reference frames used. Such is the covariance with respect to Lorentz transformations. This concept was so fruitful in the formulation of physical laws because it contains concrete chrono-geometric elements (rectilinearity and uniformity of motion) and also dynamic elements (the concept of inertia in the mechanical and the electromagnetic senses [; Section 5]). Because of this, it is related to the physical principle of relativity and itself becomes concrete and physical. If, however, in place of the Lorentz transformations one discusses arbitrary transformations, one ceases to single out that class of coordinate systems relative to which the principle of relativity exists, and by doing this one destroys the connection between physics and the concept of covariance. There remains a purely logical side to the concept of covariance as a consistency requirement on equations written in different coordinate systems. Naturally this requirement is necessary, but it is always satisfiable.

In dealing with classes of reference frames that are more general than that relative to which a principle of relativity holds, the necessity arises of replacing the explicit formulation of the principle by some other statement. The explicit formulation consists in indicating a class of physically equivalent frames of reference; the new formulation must express those properties of space and time by virtue of which the principle of relativity is possible. With the assumption of a rigid metric this is achieved by introducing an additional equation (49*.07). We saw [in this chapter] that with the additional assumption of the absence of a universal scale ($K = 0$) these equations lead to a generally covariant formulation of the theory of relativity, without any alteration of its physical content. The Galileo-Lorentz principle of relativity is then maintained to its full extent.

The very possibility of formulating the ordinary theory of relativity in a generally covariant form shows particularly clearly the difference between the principle of relativity as a physical principle and the covariance of the equations as a logical requirement. In addition, such a formulation opens the way to generalization based on a relaxation of the assumption of a rigid metric. This relaxation provides the possibility of replacing the supplementary conditions (49*.07) by others which reflect better the properties of space and time. This leads us to Einstein's theory of gravitation, which [will be discussed in the following chapters.] <u>is discussed on all spacetime scales in Appendices VI and VII.</u>

<u>APPENDIX VI</u>

(The following text is published in the Theory of Space, Time and Gravitation, Chapter V, pp. 183-209 and is incorporated herein by reference.

According to the present invention, departures from the text have been made. Additions are underlined and deletions are [bracketed] and initialed.)

THE PRINCIPLES OF THE THEORY OF GRAVITATION

50. The Generalization of Galileo's Law

The most essential characteristic of the gravitational field by which it differs from all other fields known to physics reveals itself in the effect of the field on the motion of a freely moving body of mass point. In a gravitational field all otherwise free bodies move in the same manner, provided the initial conditions of their motion, i.e. their initial position and velocities, are the same. This fundamental law may be thought of as a generalization of Galileo's law that in the absence of resistance all bodies fall equally fast.

It is appropriate to recall at this point the definitions of inertial mass and of gravitational mass. Inertial mass is the measure of the ability of a body to resist acceleration; for a given force the acceleration is inversely proportional to the inertial mass. Gravitational mass is the measure of the ability of a body to produce a gravitational field and to suffer the action of such a field; in a given field the force experienced by the body is proportional to the gravitational mass.

Using these definitions the aforementioned generalization of the Galileo's law can be formulated as a statement that the inertial and the gravitational masses af any body are equal.

The atomic basis for Galileo's law is that all matter is composed of fundamental particles comprising matter with mass that is confined to three-dimensional space-time. The two-dimensional spatial manifold of ordinary matter is a spherical shell. Such a manifold possesses positive curvature. Einstein's field equations are solved by equating the discontinuity of mass as given in the two-dimensional spatial manifold with the discontinuity of curvature of Riemannian spacetime. It will be demonstrated that the resulting gravitational equation contains the entire mass of the manifold which is also the inertial mass. Einstein's field equations are derived on the basis of this equivalence, and they are solved by matching boundary conditions at a Mills orbital. Thus, a self consistent gravitational theory valid over all scales is developed which unifies the General Theory of Relativity and the novel atomic model of the present invention.

According to Newton the gravitational field can be characterized by the gravitational potential U(x,y,z). The gravitational potential produced by an isolated spherically symmetric mass M at points exterior to itself is

$$U = \frac{\gamma M}{r} \qquad (50.01)$$

where r is the distance from the center of the mass. The quantity γ is the Newtonian constant of gravitation-in c.g.s. units it has the value

$$\gamma = \frac{1\ \text{cm}^3}{15\,000\,000\ \text{g sec}^2} \qquad (50.02)$$

Thus U has the dimensions of the square of a velocity. We note immediately that in all cases encountered in Nature, even on the surface of the Sun or of super-dense stars, the quantity U is very small compared to the square of the speed of light

$$U << c^2 \qquad (50.03)$$

In the general case of an arbitrary mass distribution the Newtonian potential U it produces satisfies Poisson's equation

$$\Delta U = -4\pi\gamma\rho \qquad (50.04)$$

where ρ is the mass density. The Newtonian potential is fully determined by Poisson's equation together with continuity and boundary conditions which are as follows: the function U and its first derivatives must be finite, single-valued and continuous throughout space and must tend to zero at infinity.

Let us assume that the Newtonian potential U is given. The force experienced by a body (mass point) of gravitational mass $(m)_{gr}$ in the gravitational field of potential U is

$$F = (m)_{gr} \text{grad} U \qquad (50.05)$$

On the other hand, by Newton's laws of motion, we have

$$(m)_{in} w = F \qquad (50.06)$$

Therefore

$$(m)_{in} w = (m)_{gr} \text{grad } U \qquad (50.07)$$

By the generalization of Galileo's law the motion of the body in a given gravitational field cannot depend on its mass. Therefore, the ratio of inertial mass $(m)_{in}$ to the gravitational mass $(m)_{gr}$ must be the same for all bodies; it is thus a universal constant whose value can only depend on the choice of units for the two masses. In the units quite generally accepted one has simply

$$(m)_{in} = (m)_{gr} = m \qquad (50.08)$$

so that the inertial and gravitational masses are equal.

The equality of inertial and gravitational mass is such a familiar fact that it is usually accepted as something obvious. However, the matter is not so simple: their equality is a separate and very important law of Nature, closely connected with the generalization of Galileo's law.

As a result of the equality of inertial and gravitational mass the equation of motion

$$w = \text{grad } U \qquad (50.09)$$

has universal character, and thus formally expresses the generalization of Galileo's law.

We note that the equations of motion (50.09) can be obtained from the variational principle

$$\delta \int \left(\frac{v^2}{2} + U\right) dt = 0 \qquad (50.10)$$

This fact will be a guide to us in constructing the theory of gravitation.

51. The Square of the Interval in Newtonian Approximation

The phenomenon of universal gravitation forces us to widen the framework of the theory of space and time [which was the subject of the foregoing chapters.] The necessity of this widening becomes clear from the following considerations.

It follows from the equation of wave front propagation, which can be stated in the form

$$\frac{1}{c^2}\left(\frac{\delta\omega}{\delta t}\right)^2 - \left[\left(\frac{\delta\omega}{\delta x}\right)^2 + \left(\frac{\delta\omega}{\delta y}\right)^2 + \left(\frac{\delta\omega}{\delta z}\right)^2\right] = 0 \qquad (51.01)$$

that light is propagated in straight lines. But light possesses energy and by the law of proportionality of mass and energy all energy is indissolubly connected with mass. Therefore, light must possess mass. On the other hand, by the law of universal gravitation, any mass located in a gravitational field must experience the action of that field and in general its motion will therefore not be rectilinear[1]. Hence it follows that in a gravitational field the law of of wave front propagation must have a form somewhat different from the one given above. But the equation of wave front propagation is a basic characteristic of the properties of space and time. Hence it follows that the presence of the gravitational field must affect the properties of space and time and their metric is then not a rigid one. This is indeed the conclusion reached in the theory of gravitation which we now begin to construct.

[As was shown in Chapter I the] The equation of wave front propagation (51.01), with some additional assumptions, leads to the following expression for the square of the interval:

$$ds^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2) \quad (51.02)$$

The influence of the gravitational field on the properties of space and time must have the consequence that the coefficients in the equation of wave front propagation and in the expression for the square of the interval will differ from the constant values appearing in (51.01) and (51.02). We must now find an approximate form for the square of the interval in a gravitational field of Newtonian potential U, relying on the generalization of Galileo's law to guide us. The fundamental fact that the law of motion for a body moving freely in a gravitational field is a universal one which does not depend on the nature of the body permits us to find the relation between the law of motion and the metric of space-time.

The equations of a geodesic in a space-time with given metric [were studied in Section 38] are given in Appendix IV. We shall now try to find a metric such that these equations coincide approximately with the Newtonian equations of motion for a free body in a given gravitational field. If this attempt is successful it will enable us to introduce the hypothesis that in a space-time with given metric a free body (mass point) moves along a geodesic; in this way the connection between the law of motion and the metric will be established.

As we know, the equation of a geodesic may be derived from the variational principle

$$\delta \int ds = 0 \quad (51.03)$$

If the squared interval is of the form (50.02) we have

$$ds = \sqrt{(c^2 - v^2)}dt \quad (51.04)$$

or, for small velocities,

$$ds = \left(c - \frac{v^2}{2c}\right)dt \quad (51.05)$$

Inserting (51.04) or (51.05) into (51.03) gives equations that describe motion with constant velocity, which indeed is free motion in the absence of a gravitational field. We can now assume that for small velocities and weak gravitational fields ($U \ll c^2$) the expression for the interval takes the form

$$ds = \sqrt{(c^2 - 2U - v^2)}dt \quad (51.06)$$

---

[1] The theory of the deflection of light in a gravitational field was given in Appendix I. Also see Appendix IV for an expossition that a wavelenghtis a two-dimensional manifold, a Mills orbital.

or

$$ds = \left[c - \frac{1}{c}\left(\frac{1}{2}v^2 + U\right)\right]dt \qquad (51.07)$$

† The theory of the deflection of light in a gravitational field [is given in Section 59 below.] was given in Appendix I. Also see Appendix IV for an exposition that a wavefront is a two-dimensional manifold, a Mills orbital.

in place of (51.04) or (51.05). Since neither an additive constant nor a constant multiplier are of any importance in the Legrangian the variational principle (51.03), with ds taken from (51.07), gives the same result as the variational principle

$$\delta \int \left(\frac{v^2}{2} + U\right)dt = 0 \qquad (51.08)$$

which was formulated at the end of Section 50, but this did indeed describe free motion of a body in a gravitational field.

It is true that just because additive constants and multiplicative factors in a Lagrangian are immaterial equation (51.08) could be obtained from (51.03) and (51.06) with any sufficiently large value of the constant c; but we must require that in the absence of gravitation, when U = 0, the expression (51.06) for the interval shall go over into the Galilean form (51.04) whatever the value of v^2. This requirement fixes the constant c in (51.06) to be equal to the speed of light.

These arguments give us good reason to assume that under conditions

$$U \ll c^2$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = v^2 \ll c^2 \qquad (51.09)$$

the square of the interval differs little from the form

$$ds^2 = (c^2 - 2U)dt^2 - (dx^2 + dy^2 + dz^2) \qquad (51.10)$$

Here the relative error in the coefficient of dt^2 will certainly be of higher order than the term $2U/c^2$ which is included. As regards the coefficient of the purely spatial part of the interval, it may differ from unity by a quantity of the same order as $2U/c^2$. Indeed, the theory of gravitation to be developed in the following sections gives the more exact expression

$$ds^2 = (c - 2U)dt^2 - \left(1 + \frac{2U}{c^2}\right)(dx^2 + dy^2 + dz^2) \qquad (51.11)$$

Under the conditions (51.09) the difference between (51.10) and (51.11) is negligible, as it should be.

In principle the value found for the coefficient of dt^2 is capable of experimental verification.

Let us assume that at some point (x_1, y_1, z_1) at which the gravitational potential is U_1, there is some emitter of radiation of proper period T_0. The wave radiated by it will have a time dependence of the form

$$\exp\left(i2\pi \frac{t}{T_1}\right) \qquad (51.12)$$

where T_1 is not equal to T_0 but is related to it in the same way as dt is related to $d\tau$, the differential of the proper time of the emitter. If, for simplicity, the emitter is assumed to be at rest in the frame of reference chosen, we have approximately

$$d\tau = \frac{1}{c} ds = \left(1 - \frac{U_1}{c^2}\right) dt \qquad (51.13)$$

and therefore

$$T_0 = \left(1 - \frac{U_1}{c^2}\right) T_1 \qquad (51.14)$$

In this problem the dependence of the gravitational potential on time may be ignored, so that the gravitational field can be treated as static. Then the wave being propagated from the emitter will retain its time dependence (51.12) throughout space.

Let us assume further that at some other point (x_2, y_2, z_2), where the gravitational potential on time may be ignored, so that the gravitational field has a different value U_2, there is a second identical emitter, e.g. another atom of the same element. The wave emitted by it will have a time dependence throughout space of the form

$$\exp\left(i2\pi \frac{t}{T_2}\right) \qquad (51.15)$$

where

$$T_0 = \left(1 - \frac{U_2}{c^2}\right) T_2 \qquad (51.16)$$

Thus the two waves emitted by identical sources but originating from places of different gravitational potential have periods differing by

$$T_2 - T_1 = \frac{U_2 - U_1}{c^2} T_0 \qquad (51.17)$$

If U_2 is the potential on the Sun and U_1 is the potential on the Earth we have $U_2 > U_1$ and the numerical value of the factor T_0 in (51.17) is approximately equal to

$$\frac{U_2 - U_1}{c^2} = 2 \times 10^{-6} \qquad (51.18)$$

Thus the wave lengths of spectral lines originating on the Sun must be displaced relative to the corresponding lines produced on the Earth by two parts in a million towards the red end of the spectrum.

However, one must note that the emission of the spectral lines on the Sun takes place in physical conditions different from those on Earth and that the change of period due to the difference of gravitational potentials is to a great extend masked by other corrections.

There are, however, certain super-dense stars, such as the companion of Sirius, which have a density tens of thousands of times greater than that of water. On their surfaces the value of gravitational potential is significantly grater than on the surface of the Sun–twenty times greater in the case of the companion of Sirius–for such stars the correction due to the difference in gravitational potential becomes very appreciable and can be detected experimentally.

[Recently, (1959)] Also, the influence of the gravitational potential on the frequency of emitted light was successfully revealed in terrestrial conditions by making use of the Mossbauer phenomenon as described in Appendix I.[ ]

52. Einstein's Gravitational Equations

Einstein's theory of gravitation in its restricted, non-cosmological, form has the following basic idea.

The geometrical properties of real physical space and time correspond not to Euclidean but Riemannian geometry [In Chapter III we discussed the basic postulates of this geometry]. Any derivation of geometrical properties from their Euclidean, or to be precise, pseudo-Euclidean form appears in Nature as a gravitational field. The geometrical properties are indissolubly linked with the distribution and motion of ponderable matter. This relationship is mutual. On the one hand, the derivations of geometrical properties from the Euclidean are determined by the presence of gravitating masses, on the other, the motion of masses is determined by these derivations. In short, masses determine the geometrical properties of space and time, and these properties determine the movement of the masses.

We shall now attempt to formulate these ideas mathematically.

In the previous section we saw that in a certain coordinate system, which for practical purposes coincides with an inertial frame of Newtonian mechanics, the Newtonian potential of gravitation U enters the coefficient of dt^2 in the expression for the square of the interval, i.e. the coefficient g_{00} of the general expression

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu \qquad (52.01)$$

On the other hand, in Newtonian approximation the gravitational potential U satisfies Poisson's equation

$$\Delta U = -4\pi\gamma\rho \qquad (52.02)$$

The required generalization of Newton's theory of gravitation must be covariant with respect to arbitrary coordinate transformations. Therefore, it is impossible to regard as a generalization of the Newtonian potential a term in the coefficient g_{00} or this whole coefficient; instead the whole set of coefficients $g_{\mu\nu}$ must be taken into consideration and must appear as the generalization of Newton's potential. The fundamental metric tensor must satisfy a set of equations that are generally covariant and in the Newtonian approximation one of them must go over into Poisson's equation for the potential U. The total number of equations must, generally speaking, be equal to the number of unknown functions, i.e. to the number of components of the tensor $g_{\mu\nu}$, which is ten.

On the left-hand side of Poisson's equation there is a second order differential operator, the Laplace operator, acting on U. Therefore, the simplest generally covariant generalization of this left-hand side will be a tensor which involves linearly the second derivatives of the metric tensor $g_{\mu\nu}$.

Such tensors are the curvature tensors (either of second or fourth rank). The fourth rank curvature tensor $R_{\mu\nu,\alpha\beta}$ is unsuitable because its components do not contain expressions which could be generalizations of the Laplace operator acting on U. Also has too many components, the number being twenty, twice as many as there are unknown functions2. Therefore, there remains the second rank curvature tensor which has just the right number of components.

On the right-hand side of Poisson's equation the mass density ρ appears. A generalization of the mass density which has the required tensor character is the mass tensor $T^{\mu\nu}$ whose invariant is equal to the invariant mass density.

We are thus led to the conclusion that the required generalization of Poisson's equation must be a relation between the second rank curvature tensor $R^{\mu\nu}$ and the mass tensor $T^{\mu\nu}$.

[In the previous chapters we saw that in] In the absence of gravitational fields the divergence of the tensor $T^{\mu\nu}$ must vanish

$$\nabla_\nu T^{\mu\nu} = 0 \qquad (52.03)$$

[We shall retain this equation for the general case, postponing the discussion of questions connected with it (questions of the energy of a gravitational field, of the integral form of conservation laws, etc.) until Chapter VII.

But we establish at the end of Chapter III that] <u>the</u> tensor

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \qquad (52.04)$$

which is known as Einstein's tensor, or the conservative tensor, has the remarkable property that its divergence is identically zero.

$$\nabla_\nu G^{\mu\nu} \equiv 0 \qquad (52.05)$$

Therefore, if we put

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -x T^{\mu\nu} \qquad (52.06)$$

where x is a constant, equations (52.03) for the mass tensor will be a consequence of (52.06).

As we know, the metric tensor $g_{\mu\nu}$ itself also satisfies (52.05); therefore, we could add to the conservative tensor on the left-hand side of (52.06) a tensor of the form lgmn where λ is a constant , without violating (52.03).

[We saw in Section 31 that if] <u>If</u> one imposes purely local conditions on the mass tensor (the condition that it should depend on the field components or other functions of state, but not on the coordinates; the vanishing of its divergence by virtue of the field equations) it is only determined apart from two constants. To be more precise, if the tensor $T^{\mu\nu}$ satisfies the conditions stated, these are also satisfied by

$$T'^{\mu\nu} = \alpha T^{\mu\nu} + \beta g^{\mu\nu} \qquad (52.07)$$

Here the constant a depends on the choice of energy unit, the constant β also on the conditions at infinity. If in supplementation of the local conditions one demands

---

[2] [It is true that even an excessive number of equations for the tensor Rmn,ab might prove to be compatible as is the case for a space of constant curvature (equation (49.12)) but in that case the equations permit the metric to be determined purely locally, i.e. without using boundary conditions. They therefore have a character different from that of Poisson's equation for which boundary conditions are essential.]

that at infinity , where the field vanishes, the mass tensor should also be zero, then β will be zero and the mass tensor is determined uniquely.

Replacement of the tensor $T^{\mu\nu}$ by the linear function (52.07) of itself corresponds to replacing the gravitational equations (52.06) by.

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -xT^{\mu\nu} - \lambda g^{\mu\nu} \qquad (52.08)$$

The constant λ is called the cosmological constant. It is clear from these remarks that the question of the value of λ acquires a definite meaning only after the conditions are formulated by which the tensor $T^{\mu\nu}$ is to be defined uniquely. Such condition will necessarily be of non-local character; they can therefore only be formulated starting from definite assumptions about the character of space-time as a whole.

At the beginning of this section we stated the basic assumptions of Einstein's theory of gravitation in its limited (non-cosmological) formulation. According to these the concept of spatial infinity retains its meaning, space-time at infinity being pseudo-Euclidean (Galilean). Deviations from Euclidean character are observed only at a finite distance from massive bodies. But in this case the mass tensor may continue to be subject to the requirement stated for the case when the whole of space-time was assumed pseudo-Euclidean. We can demand that at infinity, where the field vanishes, the mass tensor should also become zero. It then has a uniquely defined meaning to inquire after the value of the cosmological constant and the answer can be based on the following consideration. According to our basic postulates, the absence of a gravitational field signifies the absence of deviations of the geometry of space-time from the Euclidean, and therefore also the vanishing of the curvature tensor $R^{\mu\nu}$ and of its invariant R. On the other hand, the gravitational field will be absent if the tensor $T^{\mu\nu}$ is zero everywhere. Therefore, the equations $T^{\mu\nu} = 0$ and $R^{\mu\nu} = 0$ must certainly be compatible and this is only possible if the equations relating to $T^{\mu\nu}$ do not contain the term $\lambda g^{\mu\nu}$ (i.e. if $\lambda = 0$).

Thus, given the assumptions formulated at the beginning of this section and given our definition of the mass tensor, the appropriate generalization of Poisson's equation for the potential will just be equations (52.06).

As for equations (52.08), they should be used if the problem is stated cosmologically, in which case the concept of spatial infinity is inapplicable and the tensor $T^{\mu\nu}$ contains an unknown constant β, even after units have been fixed. According to the value of this constant the value of the so-called cosmological constant λ must be selected; it is evidently related to β. The choice of some particular value of λ for a given normalization of $T^{\mu\nu}$ represents a special hypothesis, which must be introduced explicitly; this is true also for the value $\lambda = 0$.

According to the novel atomic model of the present invention, the energy of a vacuum is zero; thus, the cosmological constant of the model is zero. This is the exact experimental determined value. The model unifies atomic theory and gravitation on a cosmological scale. It is demonstrated in Appendix VII that the solution of Einstein's field equations as a boundary value problem if a Mills orbital unifies atomic theory and the General Theory of Relativity on a atomic level to provide a completely unified gravitational theory for spacetime of all scales.

Returning to the non-cosmological case and to equations (52.05) we may assert that under the conditions stated (correspondence with Poisson's equation, general covariance, linearity in the second derivatives of the $g^{\mu\nu}$, identical vanishing of the left-hand side of (52.05) and Euclidean character in absence of masses) these equations are unique.

The equations (52.06) are called Einstein's gravitational equations; they play a fundamental part in the theory of gravitation. They will be examined in the following sections.

53. The Characteristics of Einstein's Equations. The Speed of Propagation of gravitation.

We begin our discussion of Einstein's gravitational equations

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -x T^{\mu\nu} \qquad (53.01)$$

with the derivation of the first order equation for their characteristics. From a physical point of view the equation of the characteristics represents the propagation law for the wave front of a gravitational wave.

Multiplying (53.01) by $g_{\mu\nu}$ and summing we obtain the relation

$$R = xT \qquad (53.02)$$

connecting the invariants of the curvature tensor and of the mass tensor. This relation enables us to write the gravitational equations in the form

$$R^{\mu\nu} = -x(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T) \qquad (53.03)$$

The contravariant curvature tensor $R_{\mu\nu}$ is expressible as

$$R^{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \frac{\delta^2 g^{\mu\nu}}{\delta x_\alpha \delta x_\beta} - \Gamma^{\mu\nu} + \Gamma^{\mu,\alpha\beta} \Gamma^{\nu}{}_{\alpha\beta} \qquad (53.04)$$

where $\Gamma^{\mu,\alpha\beta}$ is the quantity obtained from $\Gamma^{\mu}{}_{\alpha\beta}$ by raising suffixes

$$\Gamma^{\mu,\alpha\beta} = g^{\alpha\rho} g^{\beta\sigma} \Gamma^{\mu}{}_{\rho\sigma} \qquad (53.05)$$

Therefore, the last term in (53.04) does not involve second derivatives but is a homogeneous quadratic form in the $\Gamma^{\nu}{}_{\alpha\beta}$ and hence also in the first derivatives of the metric tensor.

Second derivatives appear in the first term and also in the $\Gamma^{\mu\nu}$, but the latter dependence is only through first derivatives of the quantities

$$\Gamma^{\nu} = g^{\alpha\beta} \Gamma^{\nu}{}_{\alpha\beta} \qquad (53.06)$$

[which we introduced in Section 41. We recall that the] The d'Alembertian of any function ψ may be written in the form

$$\Box \psi = g^{\alpha\beta} \frac{\delta^2 \psi}{\delta x_\alpha \delta x_\beta} - \Gamma^{\nu} \frac{\delta \psi}{\delta x_\nu} \qquad (53.07)$$

or, alternatively, as

$$\Box\,\psi = -\frac{1}{\sqrt{(-g)}}\frac{\delta}{\delta x_\beta}\left(\sqrt{(-g)}\,g^{\alpha\beta}\frac{\delta\psi}{\delta x_\alpha}\right) \qquad (53.08)$$

whence

$$\Gamma^\alpha = -\frac{1}{\sqrt{(-g)}}\frac{\delta}{\delta x_\beta}\{\sqrt{(-g)}\,g^{\alpha\beta}\} \qquad (53.09)$$

and also

$$\Gamma^\alpha = -\,\Box\,x_\alpha \qquad (53.10)$$

The $\Gamma^{\mu\nu}$ are obtained from the Γ^ν by the rule which is formally identical with the rule for forming the symmetrical contravariant derivative of a vector:

$$\Gamma^{\mu\nu} = \frac{1}{2}(\nabla^\mu\Gamma^\nu + \nabla^\nu\Gamma^\mu) \qquad (53.11)$$

or in detail

$$\Gamma^{\mu\nu} = \frac{1}{2}\left(g^{\mu\alpha}\frac{\delta\Gamma^\nu}{\delta x_\alpha} + g^{\nu\alpha}\frac{\delta\Gamma^\mu}{\delta x_\alpha} - \frac{\delta g^{\mu\nu}}{\delta x_\alpha}\Gamma^\alpha\right) \qquad (53.12)$$

of course, since Γ^ν is not a vector the $\Gamma^{\mu\nu}$ are not a tensor. This circumstance proves very useful in simplifying Einstein's equations.

Einstein's equations are generally covariant and therefore permit coordinate transformations involving four arbitrary functions. Suppose the equations are solved in some arbitrary system of coordinates. We can then go over to other coordinates by taking as independent variables four solutions of the equation $\Box\,\psi=0$. These solutions may be chosen in such a way as to satisfy the inequalities to which the $g^{\mu\nu}$ must be subject, according to Appendix IV, they may also be subjected to some additional conditions. But as long as each of the coordinates x_0, x_1, x_2, and x_3 satisfies the equation $\Box\ x_\alpha = 0$ we shall have in that system

$$\Gamma^\alpha = 0 \qquad (53.13)$$

and therefore also

$$\Gamma^{\mu\nu} = 0 \qquad (53.14)$$

We shall call such a coordinate system harmonic. At the moment we are not interested in the question of the uniqueness of the harmonic coordinate system or rather in the additional conditions which could guarantee uniqueness. Here it is important to note that the equations (53.13) are compatible with Einstein's equations and that they do not impose any essential limitation on the solutions of the latter, serving only to narrow down the class of permissible coordinate systems.

Under the conditions (53.13) the expression of the $R^{\mu\nu}$ simplifies, becoming

$$R^{\mu\nu} = -\frac{1}{2}g^{\alpha\beta}\frac{\delta^2 g^{\mu\nu}}{\delta x_\alpha \delta x_\beta} + \Gamma^{\nu,\alpha\beta}\Gamma^\nu{}_{\alpha\beta} \qquad (53.15)$$

Hence second derivatives only appear combined in the d'Alembert operator acting on the single quantity $g^{\mu\nu}$ which has the same indices as the particular $R^{\mu\nu}$ on the left-hand side.

The form of the equation of the characteristics for any given system of equations depends only on the terms containing the highest occurring order of derivatives. In the case of the system (53.01) and (53.13) these terms are just those involving the d'Alembertian.

Therefore, the system of equations of gravitation will have the same characteristics as d'Alembert's equation,

$$\Box \psi = 0 \qquad (53.16)$$

and these are easily found [A shown in Appendix C]. They have the form

$$g^{\mu\nu} \frac{\delta\omega}{\delta x_\mu} \frac{\delta\omega}{\delta x_\nu} = 0 \qquad (53.17)$$

where $\omega(x_0, x_1, x_2, x_3) = \text{const.}$ (53.18)

is the equation of a wave front, i.e. the equation of a moving surface on which any discontinuities of the gravitational field must lie.

The equation (53.17) for the propagation of a gravitational wave-front is the same as the corresponding equation for the front of a light wave in empty space on which the whole theory of space and time was based[3]. Briefly one can say that gravitation is propagated with the speed of light.

That in Einstein's theory gravitation is propagated with the speed of light is a fact of fundamental significance. It shows that the assumed form of the gravitational equations is in agreement with the general postulate of the Theory of Relativity according to which there exists a limiting velocity for the propagation of all types of action, namely the velocity of light in free space. The existence of a finite propagation velocity for gravity removes the contradiction inherent in Newton's theory of gravitation with its admission of instantaneous action at a distance.

54. A Comparison with the Statement of the Problem in Newtonian Theory. Boundary Conditions

In Newton's theory of gravitation the gravitational potential satisfies the equation

$$\Delta U = -4\pi\gamma\rho \qquad (54.01)$$

and tends to zero at infinity in such a way that

$$\lim_{r\to\infty} rU = \gamma M \qquad (54.02)$$

where M is the total mass of all the bodies of the system in question and is equal to

$$M = \int \rho \, dx \, dy \, dz \qquad (54.03)$$

Einstein's theory, which is based on the gravitational equations

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -\varkappa T^{\mu\nu} \qquad (54.04)$$

---

[3] [When we derived that law from Maxwell's equations in Section 3, we assumed space-time to be Euclidean. But, according to a remark at the end of the Appendix E, the same result can be obtained without this assumption, starting from the generally covariant form of Maxwell's equations given in Section 46.]

must, in first approximation, give the same result as Newton's theory. Newtonian theory is applicable to such mass distributions for which the total mass, given by the integral (54.03) taken over all space, remains finite. This condition is in particular satisfied by any mass distribution of insular character. We use this term to describe the case that all the masses of the system studied are concentrated within some finite volume which is separated by very great distances from all other masses not forming part of the system. When these other masses are sufficiently far away one can neglect their influence on the given system of masses, which then may be treated as isolated.

In formulating Einstein's theory we should likewise start from the assumption that the mass distribution is insular. This assumption makes it possible to impose definite limiting conditions at infinity as for Newtonian theory, and so makes the mathematical problem a determined one.

Theoretically, other assumptions are admissible. For instance, one can assume a mass distribution which on the average is uniform throughout space. Such a point of view is appropriate to the study of distances so enormous that in comparison even the distances between galaxies are taken to be very small. Very little is known of the mass distribution over such great distances and therefore a theory dealing with them will necessarily be less reliable and less capable of experimental verification than the theory of smaller scale astronomical phenomena.

[The bulk of this book will be devoted to the case of insular distributions of masses. The assumption of uniform distribution will be considered only in Sections 94 and 95, where we give the theory of Friedmann-Lobachevsky space to which this assumption leads.]

We shall thus now assume that space-time is in the main Euclidean, or rather pseudo-Euclidean, and that any deviation of space-time geometry from Euclidean geometry is a result of the presence of a gravitational field. Wherever there is no gravitational field, geometry must be Euclidean. For an insular distribution of masses the gravitational fields must tend to zero at infinity and therefore we have to assume that at points far removed from the masses the geometry of space-time becomes Euclidean. However, when geometry is Euclidean there exist Galilean coordinates x, y, z and t, in terms of which the square of the interval has the form

$$ds^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2) \qquad (54.05)$$

Since experiment shows that the geometry of space-time nowhere deviates greatly from Euclidean geometry one may expect that there should exist in the whole space variables in terms of which the square of the interval deviates but little from (54.05). In the following we shall give a more precise definition of these quasi-Galilean coordinates.

We note that Newton's theory is simplest to formulate in just these Galilean coordinates, i.e. in an inertial frame of reference. Consequently Einstein's theory, which is its generalization, should be compared with it in terms of coordinates with as similar properties as possible.

Newtonian theory is non-relativistic and in passing from a relativistic theory to a non-relativistic one, it is especial to single out the speed of light as a large parameter. Therefore we shall no longer introduce the quantity c into the parameter; instead of (35.03) we shall now write

$$x_0 = t; \; x_1 = x; \; x_2 = y; \; x_3 = z \qquad (54.06)$$

Thus henceforth the variable x_0 will mean simply that time t and not ct as previously.

The expression (54.05) for the square of the interval now appears as

$$ds^2 = c^2 dx_0^2 - (dx_1^2 + dx_2^2 + dx_3^2) \qquad (54.07)$$

This must be valid at sufficiently large distance from the masses, where the geometry is Euclidean.

Comparing with the general expression

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu \qquad (54.08)$$

we find that the $g_{\mu\nu}$ must have the following limiting values at infinity

$$(g_{oo})_\infty = c^2; \qquad (g_{oi})_\infty = 0 \qquad (54.09)$$
$$(g_{ik})_\infty = -\delta_{ik} \qquad (i, k = 1, 2, 3)$$

The corresponding limiting values of the contravariant components of the metric tensor will be

$$(g^{oo})_\infty = \frac{1}{c^2};\ (g^{oi})_\infty = 0;\ (g^{ik})_\infty = -\delta_{ik} \qquad (54.10)$$

These are then to be considered as the boundary conditions on the metric tensor.

However, the number of boundary conditions so far stated is insufficient; some additional ones must be added which characterize the asymptotic behavior of the differences $g_{\mu\nu} - (g_{\mu\nu})_\infty$ at large distances from the masses.

In the previous section we saw that, at least if $\Gamma^\nu = 0$, Einstein's equations are of the type of the wave equation, because their main terms involve the d' Alembert operator. Outside the mass distribution the tensor $T^{\mu\nu}$ vanishes and the equations take on the form

$$R^{\mu\nu} = 0 \qquad (54.11)$$

where, provided $\Gamma^\nu = 0$, the tensor $R^{\mu\nu}$ has the form

$$R^{\mu\nu} = -\frac{1}{2} g^{\alpha\beta} \frac{\delta^2 g^{\mu\nu}}{\delta x_\alpha \delta x_\beta} + \Gamma^{\nu,\alpha\beta} \Gamma^\nu{}_{\alpha\beta} \qquad (54.12)$$

We assume that at large distances the differences $g^{\mu\nu} - (g^{\mu\nu})_\infty$ and their first and second derivatives tend to zero as $1/r$, where $r = \sqrt{(x_1^2 + x_2^2 + x_3^2)}$. (This assumption will be justified in the following.) Then at large distances the second term of (54.12), being a homogeneous quadratic form in the first derivatives, will tend to zero as $1/r^2$. As for the term involving the d'Alembertian, the coefficients in it can be replaced by their limiting values to the same approximation. After these simplifications we get

$$R^{\mu\nu} \cong -\frac{1}{2c^2} \frac{\delta^2 g^{\mu\nu}}{\delta x_0^2} + \frac{1}{2} \left(\frac{\delta^2 g^{\mu\nu}}{\delta x_1^2} + \frac{\delta^2 g^{\mu\nu}}{\delta x_2^2} + \frac{\delta^2 g^{\mu\nu}}{\delta x_3^2} \right) \qquad (54.13)$$

<u>It can be shown</u> [A more complete investigation of the asymptotic behavior of the $g^{\mu\nu}$ will be given in Section 87. It shows] that the asymptotic form of $g^{\mu\nu}$ is influenced by the terms of order $1/r^2$ omitted from (54.13) but that qualitatively the behavior of the

difference $g^{\mu\nu} - (g^{\mu\nu})_\infty$ will be the same as the behavior of a function ψ satisfying the wave equation

$$\frac{1}{c^2}\frac{d^2\psi}{dt^2} - \Delta\psi = 0 \qquad (54.14)$$

where Δ is the usual, Euclidean Laplace operator.

We are interested in solutions of the wave equation (54.14) which correspond to outgoing waves dying off at infinity. They have the asymptotic form

$$\psi = \frac{1}{r} f(t - \frac{r}{c}, n) \qquad (54.15)$$

where n is a unit vector with the components

$$n_x = \frac{x}{r}, \ n_y = \frac{y}{r}, \ n_z = \frac{z}{r} \qquad (54.16)$$

and f is an arbitrary function. The function f and its derivatives with respect to all its arguments are assumed finite. The argument n gives the dependence of f on the direction along which a point recedes to infinity.

Other possible solutions of the wave equations, must be discarded for physical reasons, for in our statement of the problem the system is considered to be isolated. This means that no waves impinge on it from outside, all waves have bodies of the system as their sources and, since in a system of insular type all bodies are concentrated in a finite region, all waves originate in this region and so have the asymptotic form (54.15) at large distances from the region.

The conditions that a solution of the wave equations should at large distances have the form indicated can be stated in the differential form

$$\lim\left(\frac{\delta(r\psi)}{\delta r} + \frac{1}{c}\frac{\delta(r\psi)}{\delta t}\right) = 0 \qquad (54.17)$$

This condition must hold for $r \rightarrow \infty$ and all values of $t_0 = t + r/c$ in an arbitrary fixed interval. It can be called the condition of outward radiation. It ensures the uniqueness of the solution provided it is associated with the requirement that the function ψ and its first derivatives with respect to x, y, z and t should be everywhere bounded and should die off at infinity as $1/r$.

We note that above considerations refer strictly speaking to the ordinary wave equation (54.14) and not to Einstein's equations. Therefore, the asymptotic form of the difference.

$$g^{\mu\nu} - (g^{\mu\nu})_\infty = \psi \qquad (54.18)$$

will, in fact, differ somewhat from (54.15). However, a slightly modified condition of outward radiation written in the differential form (54.17) will be valid for (54.18).

Summing up, we can say that in our statement of the problem the metric tensor must satisfy the condition of being Euclidean at infinity and also the condition of outward radiation.

55. Solution of Einstein's Gravitational Equations in First Approximation and Determination of the Constant

To compare the gravitational theories of Einstein and of Newton we must first of all determine the constant x which enters Einstein's gravitational equations

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -xT^{\mu\nu} \qquad (55.01)$$

The value of this constant can be found by comparing the expression for the square of the interval derived in Newtonian approximation in Section 51 with that obtainable by approximately solving Einstein's equations.

For the mass tensor on the right-hand side of (55.01) we may take the approximate expression corresponding to a Euclidean metric, where the mass term is given explicitly for the case of an elastic body as follows:

$$T^{00} = \rho + \frac{1}{c^2}\left(\frac{1}{2}\rho v^2 + \rho\Pi\right)$$

$$T^{0i} = \frac{1}{c}\rho v_i + \frac{1}{c^3}\left[v_i\left(\frac{1}{2}\rho v^2 + \rho\Pi\right) - \sum_{k=1}^{3} p_{ik}v_k\right] \qquad (32.34)$$

$$T^{ik} = \frac{1}{c^2}(\rho v_i v_k - p_{ik})$$

In the equations x_0 means simply the time and not ct; therefore, the notation T^{00} will presently be equal to $c^2\,T^{00}$; and T^{0i} of equations (32.34) will presently be equal to cT^{0i}, and T^{ik} is unaltered.

Thus if x0 = t the equations (32.34) become

$$c^2T^{00} = \rho + \frac{1}{c^2}\left(\frac{1}{2}\rho v^2 + \rho\Pi\right)$$

$$c^2T^{0i} = \rho vi + \frac{1}{c^2}\left\{v_i\left(\frac{1}{2}\rho v^2 + \rho\Pi\right) - \sum_{k=1}^{3} p_{ik}v_k\right\} \qquad (55.02)$$

$$c^2T^{ik} = \rho v_i v_k - p_{ik}$$

In the present approximation we must disregard the terms corresponding to the density and current energy–Umov's scalar and vector–and we write simply

$$c^2T^{00} = \rho; \quad c^2T^{0i} = \rho v_i \qquad (55.03)$$

To the same accuracy to which this is valid we may replace the invariant of the mass tensor by the value

$$T = \rho \qquad (55.04)$$

Equations (55.03) and (55.04) enable us to calculate the approximate values of the tensor components entering the right-hand side of Einstein's equations written in the form

$$R^{\mu\nu} = -x\left(T^{\mu\nu} - \frac{1}{c}g^{\mu\nu}T\right) \qquad (55.05)$$

which was given in (53.03). Using the Galilean values of the $g^{\mu\nu}$ we get

$$T^{oo} - \frac{1}{2}g^{oo}T = \frac{1}{2c^2}\rho$$

$$T^{oi} - \frac{1}{2}g^{oi}T = \frac{1}{c^2}\rho v_i \qquad (55.06)$$

$$T^{ik} - \frac{1}{2}g^{ik}T = \frac{1}{2}\rho\delta_{ik}$$

On the other hand, according to (54.13), if we use harmonic coordinates we have approximately,

$$R^{\mu\nu} = \frac{1}{2}\Delta g^{\mu\nu} - \frac{1}{2c^2}\frac{\delta^2 g^{\mu\nu}}{\delta t^2} \qquad (55.07)$$

where Δ is the usual Euclidean Laplace operator. As we shall be interested in a quasi-static solution we can discard the term involving the second time derivative. Inserting (55.07) and (55.06) into (55.05) we have

$$\Delta g^{oo} = -\frac{x}{c^2}\rho$$
$$\Delta g^{oi} = -\frac{2x}{c^2}\rho v_i \qquad (55.08)$$
$$\Delta g^{ik} = -x\rho\delta_{ik}$$

We now refer to the expression for the squared interval in the Newtonian approximation. According to (51.10) we have then

$$g_{oo} = c^2 - 2U \qquad (55.09)$$

where U is the Newtonian potential. In this approximation the remaining components of the metric tensor are to be replaced by their Galilean values. Using the relation

$$g_{oo}g^{oo} + \sum_{i=1}^{3} g_{oi}g^{io} = 1 \qquad (55.10)$$

and the fact that the products $g_{oi}g^{io}$ are very small compared to unity we can take

$$g_{oo}g^{oo} = 1 \qquad (55.11)$$

and therefore

$$g^{oo} = \frac{1}{c^2} + \frac{2U}{c^4} \qquad (55.12)$$

But Newton's potential satisfies the equation

$$\Delta U = -4\pi\gamma\rho \qquad (55.13)$$

hence

$$\Delta g^{oo} = -\frac{8\pi\gamma}{c^4}\rho \qquad (55.14)$$

Comparing this with the first equation in (55.08) we see that the two are coincident, if Einstein's gravitational constant x is related to Newton's constant g by the relation

$$x = \frac{8\pi\gamma}{c^2} \qquad (55.15)$$

The Newtonian potential U is that solution of (55.13) which satisfies the appropriate boundary conditions at infinity. As is well-known that solution can be written in the form of a volume integral:

$$U = \gamma \int \frac{\rho'}{|r - r'|} dx'\, dy'\, dz' \qquad (55.16)$$

Side by side with this Newtonian potential we introduce the functions

$$U_i = \gamma \int \frac{(\rho, v_i)'}{|r - r'|} dx'\, dy'\, dz' \qquad (55.17)$$

which satisfy

$$\Delta U_i = -4\,\pi\gamma\rho v_i \qquad (55.18)$$

and also the conditions at infinity. In analogy with the corresponding electromagnetic quantities these functions may be called gravitational vector potentials. Now the solution of (55.08) can be written as

$$g^{oo} = \frac{1}{c^2}\left(1 + \frac{2U}{c^2}\right)$$
$$g^{oi} = \frac{4}{c^4} U_i \qquad (55.19)$$
$$g^{ik} = -\left(1 - \frac{2U}{c^2}\right)\delta_{ik}$$

We have used (55.15) to eliminate x.

We note that U has the dimensions of a velocity squared and the U_i the dimensions of a velocity cubed. In estimating the orders of magnitude of quantities we can take U to be of the order q^2 and the U_i of the order q^3, where q is some speed much smaller than the speed of light.

It is now purely a matter of algebra to obtain from the contravariant components of the metric tensor its covariant components, its determinants, etc.

To simplify the algebraic manipulations we introduce a system of quantities

$$a_{ik} = -g_{ik} + \frac{g_{oi}g_{ok}}{g_{oo}}; \quad a^{ik} = -g^{ik} \qquad (55.20)$$

where i, k = 1, 2, 3. It is easy to verify that

$$\sum_{m=1}^{3} a_{im}a^{mk} = \delta_{ik} \qquad (55.21)$$

The set of quantities a_{ik} may be interpreted as the three-dimensional spatial metric tensor, but only its algebraic properties are of importance to us here.

If we put

$$a = \mathrm{Det}\, a_{ik} \qquad (55.22)$$

and therefore

$$\frac{1}{a} = \mathrm{Det}\, a^{ik} \qquad (55.23)$$

we get

$$g = -a g_{oo} \qquad (55.24)$$

It follows directly from definition (55.20) that

$$g_{oo}g^{ok} = \sum_{m=1}^{3} a^{mk} g_{mo} \qquad (55.25)$$

and also

$$g_{io} = g_{oo} \sum_{m=1}^{3} a_{ik} g^{ok} \qquad (55.26)$$

If the $g^{\mu\nu}$ have the values (55.19) it follows that

$$a^{ik} = \left(1 - \frac{2U}{c^2}\right)\delta_{ik} \qquad (55.27)$$

and therefore

$$a_{ik} = \left(1 + \frac{2U}{c^2}\right)\delta_{ik} \qquad (55.28)$$

Noting the form of g_{oo} we get

$$g_{oo} a_{ik} = c^2 \delta_{ik} \qquad (55.29)$$

with an error of order higher than U/c^2.

Hence, with the same relative error, we have

$$g_{io} = c^2 g^{io} \qquad (55.30)$$

Using the results we obtain for the covariant components of the metric tensor

$$g_{oo} = c^2 - 2U$$
$$g_{oi} = \frac{4}{c^2} U_i \qquad (55.31)$$
$$g_{ik} = -\left(1 + \frac{2U}{c^2}\right)\delta_{ik}$$

Knowing the approximate values of the g_{oi} and the g^{oi} we can now verify the accuracy to which the use of (55.11) was justified. We note that (52.26) leads to

$$g_{oo} g^{oo} = 1 - \frac{16}{c^6} \sum_{i=1}^{3} U_i^2 \qquad (55.32)$$

If the U_i are the order q3 the above expression differs from unity by quantities of order q^2/c^2. Therefore, (55.11) may be used not only in this but also in the next higher approximation in U/c^2 or v^2/c^2. We note that (52.26) leads to

$$g_{oo} g^{oo} = 1 - g_{oo} \sum_{ik=1}^{3} a_{ik} g^{oi} g^{ok} \qquad (55.33)$$

Here g_{oo} is positive and the double sum is a positive definite quadratic form, therefore we shall always have, quite rigorously

$$g_{oo} g^{oo} \leq 1 \qquad (55.34)$$

though, as we shall have seen, the deviation from unity of the left-hand side is exceedingly small.

We now state the value of the determinant g and of $\sqrt{(-g)}$ times the contravariant components of the metric tensor. We shall write

$$g^{\mu\nu} = \sqrt{(-g)} \cdot g^{\mu\nu} \qquad (55.35)$$

We then have

$$-g = c^2 + 4U \qquad (55.36)$$

and therefore

$$\sqrt{(-g)} = c + \frac{2U}{c} \qquad (55.37)$$

Hence, using (55.35)

$$g^{oo} = \frac{1}{c} + \frac{4U}{c^2}$$
$$g^{oi} = \frac{4}{c^2}U_i \qquad (55.38)$$
$$g^{ik} = -c\delta_{ik}$$

We must estimate the magnitude of the neglected terms in R^{mn} which are quadratic in the first derivatives.

These terms are of the form $\Gamma^{\mu,\alpha\beta}\Gamma^{\nu}{}_{\alpha\beta}$. To calculate approximate values of the Christoffel symbols we could use the approximate form of the metric tensor that has just been derived. However, we shall not perform these calculations here [as the quadratic terms will be determined in detail in Chapter VI where we shall solve the gravitational equations in the next approximation.] Here we only need the order of magnitude of the quadratic terms. It turns out that the terms in R^{oo} and R^{oi} will be the sixth and those in R^{ik} of the fourth order in $1/c$. In our present approximation these terms have no influence.

It remains to verify whether the conditions which ensure that the coordinates are harmonic,

$$\Gamma^{\nu} \equiv -\frac{1}{\sqrt{(-g)}} \cdot \frac{\delta}{\delta x_{\mu}}\{\sqrt{(-g)} \cdot g^{\mu\nu}\} = 0 \qquad (55.39)$$

are satisfied to the approximation required. Let us first make it clear to what accuracy we require them to hold. If we do not omit the terms in $\Gamma^{\mu\nu}$ in the expression (53.04) for $R^{\mu\nu}$ but instead retain them to the accuracy corresponding to the approximation (55.07) for the other terms, we obtain in place of (55.07)

$$R^{oo} = \frac{1}{2}\Delta g^{oo} - \frac{1}{2c^2}\frac{\delta^2 g^{oo}}{\delta t^2} - \frac{1}{c^2}\frac{\delta\Gamma^{o}}{\delta t}$$
$$R^{oi} = \frac{1}{2}\Delta g^{oi} - \frac{1}{2c^2}\frac{\delta^2 g^{oi}}{\delta t^2} + \frac{1}{2}\left(\frac{\delta\Gamma^{o}}{\delta x_i} - \frac{1}{c^2}\frac{\delta\Gamma^{i}}{\delta t}\right) \qquad (55.40)$$
$$R^{ik} = \frac{1}{2}\Delta g^{ik} - \frac{1}{2c^2}\frac{\delta^2 g^{ik}}{\delta t^2} + \frac{1}{2}\left(\frac{\delta\Gamma^{i}}{\delta x_k} - \frac{\delta\Gamma^{k}}{\delta x_i}\right)$$

In order that the previously neglected terms in Γ^{ν} should really be small compared to terms of the type $\Delta g^{\mu\nu}$ which were retained, it is necessary that Go should be of a higher order of smallness in c than $1/c^4$ and G^i of a higher order than $1/c^2$. These conditions are indeed satisfied. For it is directly evident from (55.38) that G^i will be of fourth order in $1/c$. As for G^0, the terms of fourth order in it are

$$\Gamma o = -\frac{4}{c^4}\left(\frac{\delta U}{\delta t} + \sum_{i=1}^{3}\frac{\delta U_i}{\delta x_i}\right) = 0 \qquad (55.41)$$

These must vanish. Therefore the equation

$$\frac{\delta U}{\delta t} + \sum_{i=1}^{3}\frac{\delta U_i}{\delta x_i} = 0 \qquad (55.42)$$

must hold. As is evident from the definition of U the U_i (either by means of differential equations with boundary conditions or in terms of volume integrals) this equation is indeed satisfied as a consequence of the relation

$$\frac{\delta \rho}{\delta t} + \sum_{i=1}^{3}\frac{\delta(\rho v_i)}{\delta x_i} = 0 \qquad (55.43)$$

which expresses the law of mass conservation in Newtonian approximation.

Thus the expressions just derived for the metric tensor satisfy to first approximation not only the gravitational equations but also the "harmonic conditions" . In addition they obviously satisfy the boundary conditions at infinity. The expression for the square of the elementary interval corresponding to the expressions derived is

$$ds^2 = (c^2 - 2U)\,dt^2 - \left(1 + \frac{2U}{c^2}\right)(dx_1^2 + dx_2^2 + dx_3^2)$$

$$+ \frac{8}{c^2}(U_1 dx_1 + U_2 dx^2 + U_3 dx_3)\,dt \qquad (55.44)$$

Usually the terms involving the products $dx_i\,dt$ are of no importance. Omitting them we get the expression

$$ds^2 = (c^2 - 2U)\,dt^2 - \left(1 + \frac{2U}{c^2}\right)(dx_1^2 + dx_2^2 + dx_3^2) \qquad (55.45)$$

which involves only the Newtonian potential. This expression has already been quoted without proof in Section 51, equations (51.11).

56. The Gravitational Equations in the Static Case and Conformal Space

The metric tensor is called static if its components do not depend on the time coordinate $x_0 = t$, so that

$$\frac{\delta g_{\mu\nu}}{\delta t} = 0 \qquad (56.01)$$

and if, in addition,

$$g_{oi} = 0 \quad (i = 1, 2, 3) \qquad (56.02)$$

It is evident from physical considerations that if several masses are present they must be in motion[4]. Therefore, a static metric tensor can only occur in the case of a single mass. In spite of limited applicability, the static case is of some physical interest, first of all because it permits a deeper insight into the analogy with the Newtonian theory of gravitation (which is also a static theory) and also because in the

[4] [The problem of the motion of a system of masses is considered in detail in Chapter VI]

static case it is easy to discuss the question of the uniqueness of the solution. Also, rigorous solutions of Einstein's equations can be found in this case.

Under the conditions (56.01) and (56.02) the time coordinate is determined uniquely, while the space coordinates permit a group of transformations among themselves. Therefore, it is natural, in this problem, to use the apparatus of three-dimensional tensor analysis and to write the gravitational equations accordingly. Three-dimensional tensor analysis can be applied either directly to the spatial part of ds^2, or else to this spatial part multiplied by some factor. Remembering the approximate form of (55.29) obtained from Einstein's equations, we introduce into the spatial part a factor inversely proportional to the factor in the time part, putting

$$ds^2 = c^2V^2dt^2 - \frac{1}{V^2}h_{ik}dx_i dx_k \qquad (56.03)$$

We shall consider the quantity V^2 to be a three-dimensional scalar and the quadratic form

$$d\sigma^2 = h_{ik}dx_i dx_k \qquad (56.04)$$

to be the squared length of the arc in a certain auxiliary space, which we shall call conformal space. Three-dimensional tensor analysis will be applied to this conformal space. As may be seen by comparing (56.03) with (55.45), in the approximation in which the latter holds we can assume the conformal space to be Euclidean and the quantity V^2 to be related to the Newtonian potential U.

Thus we shall have

$$g_{oo} = c^2V^2, \quad g_{oi} = 0, \qquad g_{ik} = -\frac{h_{ik}}{V^2}, \qquad (56.05)$$

and also

$$g^{oo} = \frac{1}{c^2V^2}, \quad g^{oi} = 0, \qquad g^{ik} = -V^2h^{ik} \qquad (56.06)$$

Here the quantities h_{ik} and h^{ik} are connected by

$$h_{ij}h^{kj} = h^k_i \quad (h^k_i = \delta^k_i) \qquad (56.07)$$

this relation being analogous to (37.18) for the $g_{\mu\nu}$. Denoting by h the determinant of the h_{ik} we easily obtain

$$\sqrt{(-g)} = \frac{c}{V^2}\sqrt{h} \qquad (56.08)$$

Therefore we have

$$\sqrt{(-g)}\cdot g^{oo} = \frac{1}{cV^4}\sqrt{h}, \qquad \sqrt{(-g)}\cdot g^{ik} = -c\sqrt{h}\cdot h^{ik} \qquad (56.09)$$

and the d'Alembert operator [(41.11)] applied to a function ψ may be written as

$$\Box\,\psi = \frac{1}{c^2V^2}\frac{\delta^2\psi}{\delta t^2} - V^2(\Delta\psi)_h \qquad (56.10)$$

where $(\Delta\psi)_h$ denotes the Laplace operator in the conformal space:

$$(\Delta\psi)_h = \frac{1}{\sqrt{h}}\frac{\delta}{\delta x_i}\left(\sqrt{h}\cdot h^{ik}\frac{\delta\psi}{\delta x_k}\right) \qquad (56.11)$$

Hence we see that spatial coordinated that are harmonic in the four-dimensional sense will also be harmonic in the three-dimensional conformal space.

We denote the four-dimensional Christoffel symbols (formed from the metric tensor $g_{\mu\nu}$) by $(GP_{\mu\nu})_g$ and the three-dimensional Christoffel symbols (formed from the metric tensor h_{ik}) by $(G_{ik})_h$. Similarly we shall attach suffixes g and h respectively to quantities that are tensors in relation to the metrics $(g_{\mu\nu})$ and (h_{ik}).

The four-dimensional Christoffel symbols and the curvature tensor can be expressed in terms of the corresponding three-dimensional quantities.

These expressions will involve derivatives of the three-dimensional scalar V, which we will denote by

$$V_k = \frac{\delta V}{\delta x_k}, \quad (V^i)_h = h^{ik}V_k \qquad (56.12)$$

The suffix h attached to the V^i will sometimes be omitted for brevity. The Christoffel symbols with purely spatial indices will be

$$(\Gamma^l{}_{ik})_g = (\Gamma^l{}_{ik})_h + h_{ik}\frac{(V^l)_h}{V} - h^l{}_k\frac{V_i}{V} - h^l{}_i\frac{V_k}{V} \qquad (56.13)$$

If one or all three indices are zero, the Chrisroffel symbols become zero:

$$(\Gamma^o{}_{oo})_g = 0, \quad (\Gamma^o{}_{ik})_g = 0 \quad (\Gamma^k{}_{io})_g = 0 \qquad (56.14)$$

Finally, if two of the indices are zero we get

$$(\Gamma^i{}_{oo})_g = c^2V^3(V^i)_h; \quad (\Gamma^o{}_{oi})_g = \frac{V_i}{V} \qquad (56.15)$$

Using the general formula

$$R^\rho{}_{\sigma,\mu\nu} = \frac{\delta\Gamma^\rho{}_{\sigma\mu}}{\delta x_\nu} - \frac{\delta\Gamma^\rho{}_{\sigma\nu}}{\delta x_\mu} + \Gamma^\alpha{}_{\sigma\mu}\Gamma^\rho{}_{\alpha\nu} - \Gamma^\alpha{}_{\sigma\nu}\Gamma^\rho{}_{\alpha\mu} \qquad (56.16)$$

we can express the four-dimensional curvature tensor in terms of a three-dimensional tensor and the covariant derivatives of the three-dimensional scalar V. Leaving out elementary, though rather tedious, calculations we obtain for the components with four spatial indices

$$(R^l{}_{i,mk})_g = (R^l{}_{i,mk})_h + h_{im}\frac{1}{V}(V^l{}_k)_h - h_{ik}\frac{1}{V}(V^l{}_m)_h$$
$$+ h^l{}_k\frac{1}{V}(V_{im})_h - h^l{}_m\frac{1}{V}(V_{ik})_h - (h_{im}h^l{}_k - h_{ik}h^l{}_m)\frac{1}{V^2}(V_jV^j)_h \qquad (56.17)$$

where $(V_{ik})_h$ is the second covariant derivative of V with respect to the metric (h):

$$(V_{ik})_h = \frac{\delta^2 V}{\delta x_i\delta x_k} - (\Gamma^j{}_{ik})_h\frac{\delta V}{\delta x_j} \qquad (56.18)$$

and $(V^l{}_k)_h$ is the same derivative with a suffix raised:

$$(V^l{}_k)_h = h^{il}\,(V_{ik})_h \quad (56.19)$$

If only one index is zero, we have

$$(R^o{}_{i,mk})_g = 0 \quad (R^l{}_{o,mk})_g = 0 \quad (R^l{}_{i,ok})_g = 0 \quad (56.20)$$

If two indices are zero we get

$$(R^o{}_{o,mk})_g = 0 \quad (56.21)$$

and also

$$(R^o{}_{i,ok})_g = \frac{1}{V}(V_{ik})_h + \frac{2}{V2}V_iV_k - h_{ik}\cdot\frac{1}{V2}(V_jV^j)_h \quad (56.22)$$

and finally

$$(R^l{}_{o,ok})_g = c^2V^2\,\{VV^l{}_k + 2V^lV_k - h^l{}_k\,(V_jV^j)_h\} \quad (56.23)$$

It is now simple to form the contracted curvature tensor which enters Einstein's gravitational equations.

Using the formula

$$(R_{ik})_g = (R^m{}_{i,mk})_g + (R^k{}_{i,ok})_g \quad (56.24)$$

we obtain from (56.17) and (56.22)

$$(R_{ik})_g = (R_{ik})_h - h_{ik}\cdot\frac{1}{V}(\Delta V)_h + \frac{2}{V2}V_iV_k + h_{ik}\cdot\frac{1}{V}(V_jV^j)_h \quad (56.25)$$

where $(\Delta V)_h$ is the Laplace operator applied to V:

$$(\Delta V)_h = (V^k{}_i)_h = \frac{1}{\sqrt{h}}\frac{\delta}{\delta x_i}\left(\sqrt{h}\cdot h^{ik}\frac{\delta V}{\delta x_k}\right) \quad (56.26)$$

On the other hand the formula

$$(R_{oo})_g = (R^k{}_{o,ko})_g = -(R^k{}_{o,ok})_g \quad (56.27)$$

gives, using (56.23),

$$(R_{oo})_g = -c^2V^2\,\{V\cdot(\Delta V)h - (V_jV^j)_h\} \quad (56.28)$$

As for the mixed components of $R_{\mu\nu}$, they vanish as a consequence of (56.20):

$$(R_{oi})_g = 0 \quad (56.29)$$

By virtue of the relations (56.06) between the four- and three-dimensional metric tensors, equations (56.25), (56.28) and (56.29) lead to the following expression for the invariant $(R)_g$:

$$(R)_g = -V^2(R)_h + 2V\,(\Delta V)_h - 4\,(V_jV^j)_h \quad (56.30)$$

We denote by $\Gamma_{\mu\nu}$ the conservative Einstein tensor

$$G_{\mu\nu} = (R_{\mu\nu})_g - \frac{1}{2}g_{\mu\nu}(R)_g \quad (56.31)$$

and by H_{ik} the conservative tensor in the conformal space

$$H_{ik} = (R_{ik})_h - \frac{1}{2}h_{ik}(R)_h \quad (56.32)$$

The invariant of the latter is

$$H = h^{ik}H_{ik} = -\frac{1}{2}\,(R)_h \quad (56.33)$$

as a result of which we have

$$(R_{ik})_h = H_{ik} - h_{ik}H \quad (56.34)$$

For the spatial part of the conservative Einstein tensor we obtain the simple expression

$$G_{ik} = H_{ik}\frac{2}{V2} V_i V_k - h_{ik}\frac{1}{V2}(V_j V^j)_h \qquad (56.35)$$

which is remarkable for the fact that it does not contain second derivatives of the three-dimensional scalar V.

The mixed components of the conservative tensor vanish,

$$G_{io} = 0 \qquad (56.36)$$

while the component G_{00} is given by

$$G_{oo} = c^2V^2\{-V^2H - 2V(\Delta V)_h + 3(V_jV^j)_h\} \qquad (56.37)$$

We go over to the formulation of the gravitational equations. We have just noted that the quantities G_{ik} do not contain second derivatives of V. On the other hand, (56.28) shows that R_{00} does not involve second derivatives of the h_{ik}. Therefore, if we write down the gravitational equations in a form solved with respect to R_{00} and G_{ik} the second derivatives of V and of h_{ik} will be separated from each other. By virtue of the general equations

$$R_{\mu\nu} = -x(T\mu\nu - \frac{1}{2}g_{\mu\nu}T) \qquad (56.38)$$

which are the covariant form of (53.03), we have

$$R_{oo} = -x(Too - \frac{1}{2}g_{oo}T) \qquad (56.39)$$

where T is the invariant

$$T = (T)_g = (T^o{}_o + T^1{}_1 + T^2{}_2 + T^3{}_3)_g \qquad (56.40)$$

Hence by virtue of (56.05) we get

$$R_{oo} = -\frac{1}{2}xc^2V^2(T^o{}_o - T^1{}_1 - T^2{}_2 - T^3{}_3)_g \qquad (56.41)$$

and, by using the value (56.28) of R_{00}:

$$V\cdot(\Delta V)_h - (V_jV^j)_h = \frac{1}{2}x(T^o{}_o - T^1{}_1 - T^2{}_2 - T^3{}_3)_g \qquad (56.42)$$

The equations for the spatial components are

$$H_{ik} + \frac{2}{V2}V_iV_k - h_{ik}\cdot\frac{1}{V2}(V_jV^j)_h = -xT_{ik} \qquad (56.43)$$

As regards the equations for the mixed components,

$$G_{io} = -xT_{io} \qquad (56.44)$$

they are satisfied identically, because here the left-hand side is zero by virtue of (56.36) and the right-hand side is also zero, because the mass current is zero.

The equations so obtained acquire a more pictorial form if one introduces some new symbols.

We put

$$\mu = \frac{1}{V2}(T^o{}_o - T^1{}_1 - T^2{}_2 - T^3{}_3)_g \qquad (56.45)$$

The quantity m can also be written in the form

$$\mu = c^2T^{oo} + h^{ik}T_{ik} \qquad (56.45)$$

As may be seen by comparing this with the approximate expression (55.02) the quantity m represents a certain mass density; we shall see that it can be interpreted as the mass density in the conformal space.

Further, we replace V by the quantity ϕ, according to the formula

$$V = e^{-\phi} \qquad (56.46)$$

so that the relation between the space-time metric and the metric in the conformal space takes on the form

$$ds^2 = c^2e^{-2\phi}dt^2 - e^{2\phi}d\sigma^2 \qquad (56.47)$$

we have

$$\frac{\delta\phi}{\delta x_i} \equiv \phi_i = -\frac{V_i}{V} \qquad (56.48)$$

$$\Delta\phi = -\frac{\Delta V}{V} + \frac{V_j V^j}{V^2}$$

Therefore, the gravitational equations (56.42) and (56.43) may be written as

$$(\Delta\phi)_h = -\frac{1}{2}x\mu \qquad (56.49)$$

$$H_{ik} = -2\phi_i\phi_k - h_{ik}(\phi_j\phi^j)_h - xT_{ik} \qquad (56.50)$$

The first of these equations is essentially Poisson's equation for the Newtonian potential U. Indeed, the symbol Δ is a generalization of the Laplace operator, m is the mass density and, by (55.15), the constant x is given by

$$x = \frac{8\pi\gamma}{c^2} \qquad (56.51)$$

Therefore, if we put

$$\phi = \frac{U}{c^2} \qquad (56.52)$$

the quantity U will satisfy the equation

$$(\Delta U)_h = -4\pi\gamma\mu \qquad (56.53)$$

which differs from equation (55.13) for the Newtonian potential.
We can also put

$$\phi_i = \frac{g_i}{c^2} \qquad (56.54)$$

where g_i is a component of the gravitational acceleration

Let us now clarify the physical meaning of (56.50). Apart from a factor the terms involving ϕ_i can be interpreted as gravitational stresses. If we put

$$2\phi_i\phi_k - h_{ik}(\phi_j\phi^j)_h = xT^*_{ik} \qquad (56.55)$$

we can replace (56.50) by

$$H_{ik} = -x(T^*_{ik} + T_{ik}) \qquad (56.56)$$

The three-dimensional divergence of the tensor T_{ik}, understood as referring to the metric (h_{ik}) is

$$(\nabla^k T^*_{ik})_h = \frac{2}{x}\phi_i(\Delta\phi)_h \qquad (56.57)$$

and by (56.49) we have

$$(\nabla^k T^*_{ik})_h = -\mu\phi_i \qquad (56.58)$$

On the other hand, since H_{ik} is a conservative tensor in the conformal space its divergence is zero. Therefore, apart from its sign, the divergence of the gravitational stress tensor is equal to the divergence of the tensor of elastic and other static stresses T_{ik}. Thus we have

$$(\nabla^k T_{ik})_h = \mu\phi_i \qquad (56.59)$$

These equations represent a generalization of the usual equations in the statics of elastic bodies in a gravitational field.

The equations for the statics in conformal space, written in the form (56.56) stand in analogy to Einstein's equations in space-time. In both sets of equations the left-hand side involves a conservative tensor, while on the right there is a stress tensor or its four-dimensional generalization. Here the gravitational stresses appear in explicit form only after space have been split off from time and after passage to the conformal space.

The conformal space will be almost Euclidean. Indeed, as is seen from (56.54) and from the estimates (55.02) for the tensor T_{ik} the right-hand side of (56.56) will be if the order g_i^2/c^2. This leads to the result that the deviation of the h_{ik} from their Euclidean values will be of the order U^2/c^4. This result is in agreement with the approximate formula (55.45), which was just the basis for introducing the conformal space.

For empty space, when $T_{\alpha\beta} = 0$ and $\mu = 0$, equation (56.49) is a consequence of (56.50). It is easy to see this by equating the divergence of H_{ik} to zero using (56.57).

If the mass tensor $T_{\alpha\beta}$ is zero the whole of space, the only static solution of Einstein's equations which has no singular points and which satisfies the boundary conditions will be the solution corresponding to Euclidean space and pseudo-Euclidean space-time. This can be shown in the following manner. In the case of empty space, equation (56.49) gives $(\Delta\phi)_h = 0$. This is an equation of the elliptic type for f, which represents a generalization of Laplace's equation. The function f and its derivative f_i must be everywhere finite and continuous and at spatial infinity they must tend to zero. But the only solution of Laplace's equation that satisfies these conditions is the solution $\phi = 0$. But then the derivatives ϕ_i will also vanish and therefore also expression (56.55). Since in addition $T_{ik} = 0$ we also have $H_{ik} = 0$. Hence it follows that the curvature tensor of the conformal space is zero, and the space itself is Euclidean.

APPENDIX VII

(The following text is published in the Theory of Space, Time and Gravitaton, Chapter v, pp. 209-215 and is incorporated herein by reference.

According to the present invention, departures from the text have been made. Additions are underlined and deletions are [bracketed] and initialed.)

57. Rigorous Solution of the Gravitational Equations for a Single Mills Orbital and a Gravitating Mass [Concentrated Mass]

In the case of a [concentrated] mass as a Mills orbital a rigorous spherically symmetric solution of the gravitational equations can be found. As we are dealing with a static case we can use the results of the foregoing section, and write ds^2 as

$$ds^2 = c^2 V^2 dt^2 - \frac{1}{V^2} d\sigma^2 \qquad (57.01)$$

$$d\sigma^2 = h_{ik}\, dx_i\, dx_k \qquad (57.02)$$

If x_1, x_2 and x_3 are harmonic coordinates we can introduce spherical coordinates related to them by putting

$$\begin{aligned} x_1 &= r^* \sin\theta \cos\varphi \\ x_2 &= r^* \sin\theta \sin\varphi \qquad (57.03) \\ x_3 &= r^* \cos\theta \end{aligned}$$

The assumption of spherical symmetry implies that the expression for $d\sigma^2$ is of the form

$$d\sigma^2 = F^2 dr^{*2} + \rho^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) \qquad (57.04)$$

where F and ρ are functions of r^* only. The coefficient V must also be taken to depend only on r^*.

We note first of all that if we put

$$F\, dr^* = dr \qquad (57.05)$$

we can reduce the general expression (57.04) to the case $F = 1$, so that

$$d\sigma^2 = dr^2 + \rho^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) \qquad (57.06)$$

It is true that in doing this it may happen that the radius-vector r will be "non-harmonic", in the sense that it is not related to the harmonic coordinates, x_1, x_2 and x_3 by equations of the form (57.03). But having formed Laplace's equation for the quantities (57.03) with r^* replaced by r one can always go over afterwards to a "harmonic" radius vector r^*.

For the metric (57.06) we get

$$h_{rr} = 1, \quad h_{\vartheta\vartheta} = \rho^2, \quad h_{\varphi\varphi} = \rho^2 \sin^2\vartheta, \qquad (57.07)$$
$$h_{\vartheta\varphi} = 0, \quad h_{\varphi r} = 0, \quad h_{r\vartheta} = 0$$

and therefore

$$h^{rr} = 1, \quad h^{\vartheta\vartheta} = \frac{1}{\rho^2}, \quad h^{\varphi\varphi} = \frac{1}{\rho^2 \sin^2\vartheta}, \qquad (57.08)$$
$$h^{\vartheta\varphi} = 0, \quad h^{\varphi r} = 0, \quad h^{r\vartheta} = 0$$

Hence

$$\sqrt{h} = \rho^2 \sin\vartheta \qquad (57.09)$$

and the Laplace operator in the conformal space may be written as

$$\Delta\psi = \frac{1}{\rho^2}\frac{\delta}{\delta r}\left(\rho^2 \frac{\delta\psi}{\delta r}\right) + \frac{1}{\rho 2}\Delta^*\psi \qquad (57.10)$$

where $\Delta^*\psi$ is the Laplace operator on a sphere:

$$\Delta^*\psi = \frac{1}{\sin\vartheta}\frac{\delta}{\delta\vartheta}\left(\sin\vartheta \frac{\delta\psi}{\delta\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\delta^2\psi}{\delta\phi^2} \qquad (57.11)$$

As a consequence of (56.10) the harmonic coordinates must satisfy Laplace's equation in conformal space. For the quantities (57.03) we have

$$\Delta^* x_i = -2x_i \qquad (57.12)$$

Therefore the condition for harmonic coordinates

$$\Delta x_i = 0 \qquad (57.13)$$

reduces to the form

$$\frac{d}{dr}\left(\rho^2 \frac{dr*}{dr}\right) - 2r* = 0 \qquad (57.14)$$

This is the equation to be used in passing from the initial radius vector r to the harmonic one, r *.
By applying the general formulae to the metric tensor (57.07) and (57.08) the following expressions for the 18 Christoffel symbols can be derived:

$$\begin{array}{lll}
\Gamma^{r}_{rr} = 0 & \Gamma^{\theta}_{rr} = 0 & \Gamma^{\phi}_{rr} = 0 \\
\Gamma^{r}_{\theta\theta} = \rho\rho' & \Gamma^{\theta}_{\theta\theta} = 0 & \Gamma^{\phi}_{\theta\theta} = 0 \\
\Gamma^{r}_{\phi\phi} = -\rho\rho'\sin^2\theta & \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta & \Gamma^{\phi}_{\phi\phi} = 0 \\
\Gamma^{r}_{r\theta} = 0 & \Gamma^{\theta}_{r\theta} = \frac{\rho'}{\rho} & \Gamma^{\phi}_{r\theta} = 0 \\
\Gamma^{r}_{r\phi} = 0 & \Gamma^{\theta}_{r\phi} = 0 & \Gamma^{\phi}_{r\phi} = \frac{\rho}{\rho'} \\
\Gamma^{r}_{\theta\phi} = 0 & \Gamma^{\theta}_{\theta\phi} = 0 & \Gamma^{\phi}_{\theta\phi} = \cot\theta
\end{array} \qquad (57.07)$$

Here the prime denotes differentiation with respect to r . The Christoffel symbols and all tensor quantities used in this section refer only to the conformal space; therefore there is no necessity to attach a suffix, as was done in Section 56.

Using the Christoffel symbols tabulated in (57.15) we form the three-dimensional fourth rank curvature tensor and then, using the equations

$$\begin{array}{ll}
R_{rr} = R^{\theta}_{r\,\theta r} + R^{\phi}_{r\,\phi r}; & R_{r\theta} = R^{\phi}_{r\,\phi\theta}; \\
R_{\theta\theta} = R^{r}_{\theta\,r\theta} + R^{\phi}_{\theta\,\phi\theta}; & R_{r\phi} = R^{\theta}_{r\,\theta\phi}; \\
R_{\phi\phi} = R^{r}_{\phi\,r\phi} + R^{\theta}_{\phi\,\theta\phi}; & R_{\theta\phi} = R^{r}_{\theta\,r\phi};
\end{array} \qquad (57.16)$$

the second rank curvature in the conformal space. in the equations for the non-diagonal components we have omitted terms in which the first lower index is equal to the upper index: owing to the symmetry properties of the fourth rank curvature tensor these terms vanish if the coordinate system is orthogonal. In the general formula (56.16) we leave only those terms which are different from zero and so obtain

$$R^{\vartheta}_{r\,\vartheta r} = \frac{\delta\Gamma^{\vartheta}_{r\vartheta}}{\delta r} + \Gamma^{\vartheta}_{r\vartheta}\,\Gamma^{\vartheta}_{\vartheta r} \qquad (57.17)$$

and after inserting the values of the Christoffel symbols from (57.15)

$$R^{\vartheta}{}_{r\vartheta r} = \frac{\rho''}{\rho} \qquad (57.18)$$

The calculation shows that $R^{\varphi}{}_{r\varphi r}$ has the same value:

$$R^{\varphi}{}_{r\varphi r} = \frac{\rho''}{\rho} \qquad (57.19)$$

Therefore

$$R_{rr} = 2\frac{\rho''}{\rho} \qquad (57.20)$$

Further, we have

$$R^{r}{}_{\vartheta r\vartheta} = -\frac{\delta\Gamma^{r}{}_{\vartheta r}}{\delta r} + \Gamma^{\vartheta}{}_{\vartheta r}\,\Gamma^{r}{}_{\vartheta\vartheta} \qquad (57.21)$$

whence

$$R^{r}{}_{\vartheta r\vartheta} = \rho\rho'' \qquad (57.22)$$

Continuing the calculation, we get

$$R^{\varphi}{}_{\vartheta\varphi\vartheta} = -\frac{\delta\Gamma^{\varphi}{}_{\vartheta\varphi}}{\delta\vartheta} + \Gamma^{\varphi}{}_{\vartheta\varphi}\,\Gamma^{\varphi}{}_{\vartheta\varphi} - \Gamma^{r}{}_{\vartheta\vartheta}\,\Gamma^{\varphi}{}_{r\varphi} \qquad (57.23)$$

whence

$$R^{\varphi}{}_{\vartheta\varphi\vartheta} = -1 + \rho'^2 \qquad (57.24)$$

Inserting (57.22) and (57.24) into (57.16) we find for $R_{\vartheta\vartheta}$ the expression

$$R_{\vartheta\vartheta} = \rho\rho'' + \rho'^2 - 1 \qquad (57.25)$$

Similar calculations give

$$R_{\varphi\varphi} = \sin^2\vartheta R_{\vartheta\vartheta} \qquad (57.26)$$

as was to be expected for reasons of spherical symmetry. The non-diagonal elements of the second rank curvature tensor prove to be zero:

$$R_{r\vartheta} = 0; \quad R_{r\varphi} = 0; \quad R_{\vartheta\varphi} = 0 \qquad (57.27)$$

The invariant of the three-dimensional curvature tensor can be calculated from the formula

$$R = R_{rr} + \frac{2}{\rho^2} R_{\vartheta\vartheta} \qquad (57.28)$$

and will be given by

$$R = \frac{2}{\rho^2}(2\rho\rho'' + \rho'^2 - 1) \qquad (57.29)$$

Applying equation (56.32) we get the following simple expression for the conservative tensor of the conformal space

$$H_{rr} = \frac{1-\rho'^2}{\rho^2} \qquad H_{\theta\theta} = -\rho\rho'' \qquad H_{\phi\phi} = -\rho\rho''\sin^2\theta, \qquad (57.30)$$

$$H_{r\theta} = 0 \qquad H_{r\phi} = 0 \qquad H_{\theta\phi} = 0$$

{We could have obtained these expressions by a somewhat simpler method using the relation which, in three dimensional space, connects the covariant fourth rank tensor and the conservative tensor. This relation is discussed in the Appendix G. In the notation of this section equations (G.13) of the Appendix may be written in the form

$$hH^{rr} = R_{\theta\phi,\theta\phi}\ ; \quad hH^{\theta\theta} = R_{r\phi,r\phi}\ ; \quad hH^{\phi\phi} = = R_{r\theta,r\theta}\ ;$$
$$hH^{\theta\phi} = R_{\phi r,r\theta}\ ; \quad hH^{\phi r} = R_{r\theta,\theta\phi}\ ; \quad hH^{r\theta} = R_{\theta\phi,\phi r}\ ; \qquad (57.31)$$

It is easy to see that these formulae lead to expressions (57.30) as previously found.]

The formulae we have derived allow us to write down Einstein's gravitational equations in explicit form. In the previous section we saw that if one writes ds^2 in the form

$$ds^2 = c^2e^{-2\phi}\, dt^2 - e^{2\phi}d\sigma^2 \qquad (57.32)$$

where has the value (57.04), the gravitational equations appear as

$$\Delta\phi = -\frac{1}{2}x\mu \qquad (57.33)$$

$$H_{ik} = -2\phi_i\phi_k - h_{ik}\,(\phi_j\phi^j)_h - xT_{ik} \qquad (57.34)$$

where the "mass density μ is given by (56.44) and (56.45). Going over to the present case of a <u>shell having</u> mass concentrated at a radius r_0 [point] and using spherical coordinates, in which the Laplace operator has the form (57.10) while the quantities H_{ik} are given by (57.30), we obtain

$$\Delta\phi = \frac{1}{\rho^2}\frac{d}{dr}(\rho^2\phi') = 0 \qquad (57.35)$$

$$H_{rr} = \frac{1-\rho'^2}{\rho^2} = -\phi'^2,$$

$$H_{\theta\theta} = -\rho\rho'' = H_{\theta\theta} = -\rho^2\phi'^2$$

The equation for $H_{\phi\phi}$ differs from that for $H_{\theta\theta}$ only by the factor $\sin^2\theta$, while the remaining equaitons of (57.34) are satisfied identically.

Integrating (57.35) we get

$$\rho^2\phi' = -\alpha \qquad (57.37)$$

where is a constant. Since equation (57.35) is a limiting case of (57.33) with positive μ, the constant α should be taken positive. Indeed, considering forst (57.33) and putting

$$4\pi\int\mu\rho^2 dr = m \qquad (57.38),$$

<u>where the integral is extended over the whole region in which μ differs from zero which is at the radius $r = r_0$ and where</u>

$$\mu = \frac{m}{4\pi r_0^2}\,\delta\,(r - r_0).$$

<u>We see that (57.37) holds everywhere for $r \geq r_0$ with α given by</u>

$$\alpha = \frac{\gamma m}{c^2} \qquad (57.39A),$$

where m is the mass of the Mills orbital. Here μ is the Newtonian gravitational constant, related to x by (56.51). Thus, it is evident that the constant α of (57.37) is solved by matching boundary conditions of the discontinuity of curvature of spacetime for $r > r_0$. The superposition principle holds; thus, the total curvature is given as the sum of the curvatures produced by Mills orbitals, and the mass in the relationship for α (57.39A) is given as the total mass; M, of all the Mills orbitals of the gravitating body

$$\alpha = \frac{\gamma M}{c^2} \qquad (57.39)$$

$$[4\pi \int \mu \rho^2 \, dr = M \qquad (57.38)$$

where the integral is extended over the whole region in which μ differs from zero, we see that (57.370 holds everywhere outside this region, with α given by

$$\alpha = \frac{\gamma M}{c^2} \qquad (57.39)$$

Here γ is the Newtonian gravitational constant, related to x by (56.51). Evidently M is the mass of the gravitating body; in] In going over to the case of a concentrated mass this quantity, and with it α, remain finite and positive. The dimensions of α are those of a length, which is why it is called the gravitational radius of the mass. Inserting the value of ϕ' from (57.37) into the first equation of (57.36) we obtain

$$\rho'^2 = 1 + \frac{\alpha^2}{\rho^2} \qquad (57.40)$$

and taking the square root so as to satisfy the requirement that we have $\rho' \; + 1$ at infinity, we get

$$\rho \rho' = \sqrt{(\rho^2 + \alpha^2)} \qquad (57.41)$$

Differentiating this expression with respect to r we obtain

$$\rho \rho' + \rho'^2 = 1 \qquad (57.42)$$

which shows that the second equation of (57.36) is also satisfied.

The differential equation (57.41) is easy to solve by quadrature; after setting the additive constant zero, we get

$$r = \sqrt{(\rho^2 + \alpha^2)} \qquad (57.43)$$

whence

$$\rho = \sqrt{(r^2 - \alpha^2)} \qquad (57.44)$$

Thus finally

$$d\sigma^2 = dr^2 + (r^2 - \alpha^2)(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) \qquad (57.45)$$

By its physical nature ρ must be postive and therefore the range of variation of r is

$$r \geq \alpha \qquad (57.46)$$

We must now discuss the harmonic condition. Inserting the value of ρ from (57.44) into (57.14) we see that the harmonic radius vector r^* satisfies the equation

$$\frac{d}{dr}(r^2 - \alpha^2)\frac{dr*}{dr} - 2r* = 0 \qquad (57.47)$$

Evidently this equation has the solution

$$r^* = r \qquad (57.48)$$

It is easy to show that this is uniquely the solution which for finite r remains finite in the whole region (57.46) and which at infinity differs from r by not more than a finite quantity. Therefore the variable r which enters our formulae is itself the harmonic radius vector and in place of (57.03) we can simply write

$$x_1 = r \sin\theta \cos\varphi$$
$$x_2 = r \sin\theta \sin\varphi \qquad (57.49)$$
$$x_3 = r \cos\theta$$

It remains to find the quantity ϕ. Integrating (57.37) and taking into account the boundary conditions we get from (57.44)

$$\phi = \int_r^\infty \frac{\alpha\supseteq dr}{r^2 - \alpha^2} \qquad (57.50)$$

or

$$\phi = \frac{1}{2}\ln\frac{r+\alpha}{r-\alpha} \qquad (57.51)$$

Hence

$$V^2 = \frac{r-\alpha}{r+\alpha} \qquad (57.52)$$

The expression (57.01) or (57.32) for $ds2$ takes on the form

$$ds^2 = c^2\frac{r-\alpha}{r+\alpha}dt^2 - \frac{r+\alpha}{r-\alpha}d\sigma^2 \qquad (57.53)$$

and, after inserting the value of $ds2$ from equation (57.45) we get

$$ds^2 = c2\left(\frac{r-\alpha}{r+\alpha}\right)dt^2 - \left(\frac{r+\alpha}{r-\alpha}\right)dr^2 - (r+\alpha)^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \qquad (57.54)$$

The rigorous solution so obtained confirms our conclusion that the conformal space is almost Euclidean and that ϕ is approximately equal to $U/c2$, where U is the Newtonian potential for which we can put

$$U = \frac{\gamma M}{r} \qquad (57.55)$$

Indeed, equation (57.45) shows that the components of the metric tensor of $ds2$ have relative deviations from their Euclidean values of the order

$$\frac{\alpha^2}{r^2} = \frac{U^2}{c^4} \qquad (57.56)$$

and that the error of replacing ϕ by $U/c2$ will be of the same order. [As the estimate given in the following section will show,] the quantity (57.56) is extremely small. One should note that such close agreement with Newton's theory is obtained only if harmonic coordinates are used.

[The solution of the problem of a concentrated mass in a form equivalent to (57.54), but in arbitrary non-harmonic coordinates, was first derived by Schwarzschild [18] and is often named after him.]

Modification and substitution mode by one of ordinary skill in the art are within the scope of the present invention, which is not limited except by the claims which follow:

CLAIMS

What is claimed is:

1. A method of providing a repulsive force from a gravitating mass having a spacetime manifold of a first curvature comprising the steps of:

providing an element of matter;

forming said element of matter into a second curvature opposite to said first curvature;

applying energy from an energy source to said element of matter having a second curvature wherein a repulsive force away form said gravitating mass is created;

receiving said repulsive force on said energy source from said element of matter in response to the force provided by said gravitating mass and said applied energy.

2. The method of claim 1, wherein said step of providing an element of matter comprises the step of providing an electron.

3. The method of claim 1, wherein said first curvature comprises a positive curvature.

4. The method of claim 3, wherein the step of forming comprises the step of

applying one of a quadrapole electrostatic field, a quadrapole magnetic field, and an electromagnetic field, and further including the step of

moving said electron through said selected one of the quadrapole electrostatic field, quadrapole magnetic field, and electromagnetic field.

5. he method of claim 4, wherein the step of moving includes the step of containing said electron.

6. The method of claim 1, wherein the step of applying energy comprises the step of applying at least one of electrostatic, magnetostatic and electromagnetic energy.

7. The method of claim 6, further including the step of applying the received repulsive force to a structure movable in relation to said gravitating means.

8. The method of claim 7, further including the step of rotating said structure around an axis providing a force having an angular momentum vector of said circularly rotating structure parallel to the central vector of the gravitational force by said gravitating mass.

9. he method of claim 8, further including the step of changing the orientation of said angular momentum vector to accelerate said structure through a trajectory parallel to the surface of said gravitating mass.

10. A method of providing a repulsive force form a gravitating mass having a space time manifold of a first curvature comprising the steps of:

providing an element of matter having a second curvature opposite to said first curvature;

applying energy form an energy source to said element of matter having a second curvature wherein a repulsive force away from said gravitating mass is created;

receiving said repulsive force on said energy source from said element of matter in response to the force provided by said gravitating mass and said applied energy.

11. Apparatus for providing repulsion from a gravitating body having a spacetime manifold of a first curvature comprising:

an element of matter;

means for forming said element of matter in a curvature opposite to the curvature of said gravitating body; and

means for applying energy to said oppositely curved element of matter, wherein

a repulsive force developed by said oppositely curved element of matter in response to said applied energy and is impressed on said means for applying energy in a direction away from said gravitating body.

12. The apparatus of claim 11, wherein said element of matter comprises an electron.

13. The apparatus of claim 12, wherein said means for forming comprises

means for providing a negatively curved field on said electron, and

means for moving said electron through said negatively curved field, causing said electron to assume a negative curvature.

14. The apparatus of claim 13, wherein said means for providing a negatively curved field comprises means for providing at least one of a quadrapole electrostatic field, a quadrapole magnetic field and a quadrapole electromagnetic field.

15. The apparatus of claim 13, wherein said means for moving includes at least one of

means for accelerating said electron along a selected path; and

means for containing said electron along a selected path.

16. The apparatus of claim 15,.wherein said means for accelerating comprises means for providing at least one of an electric field gradient, a magnetic field and an electromagnetic field along said selected path.

17. The apparatus of claim 15, wherein said means for containing comprises at least one of a means for providing spatially and temporally coherent light along said selected path, a means for providing an electric field and a means to provide a magnetic field.

18. The apparatus of claim 11, further including

a circularly rotatable structure having a moment of inertia; and

means for applying said repulsive force to circulating rotatable structure, wherein

the angular momentum vector of said circularly rotatable structure is parallel to the central vector of the gravitational force produced by said gravitating body.

19. The apparatus of claim 18, wherein said accelerating force is selectively applied to provide a trajectory parallel to the surface of said circularly rotatable structure around said gravitating body.

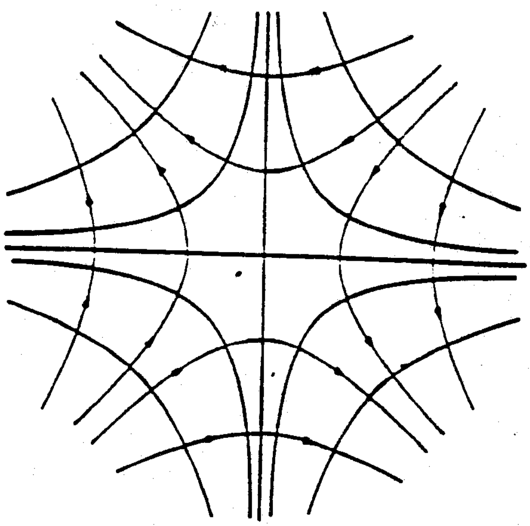
20. The apparatus of claim 18, wherein said accelerating force is selectively applied to provide a hyperbolic trajectory of the circularly rotatable structure around said gravitating body.

21. Apparatus for providing a repulsion from a gravitating body having a spacetime manifold of a first curvature comprising:

an element of matter having a curvature opposite to the curvature of said gravitating body; and means for applying energy to said oppositely curved element of matter, wherein

a repulsive force is developed by said oppositely curved element of matter in response to said applied energy and is impressed on said means for applying energy in a direction away form said gravitating body.

Figure 1.



2/8

Figure 2.

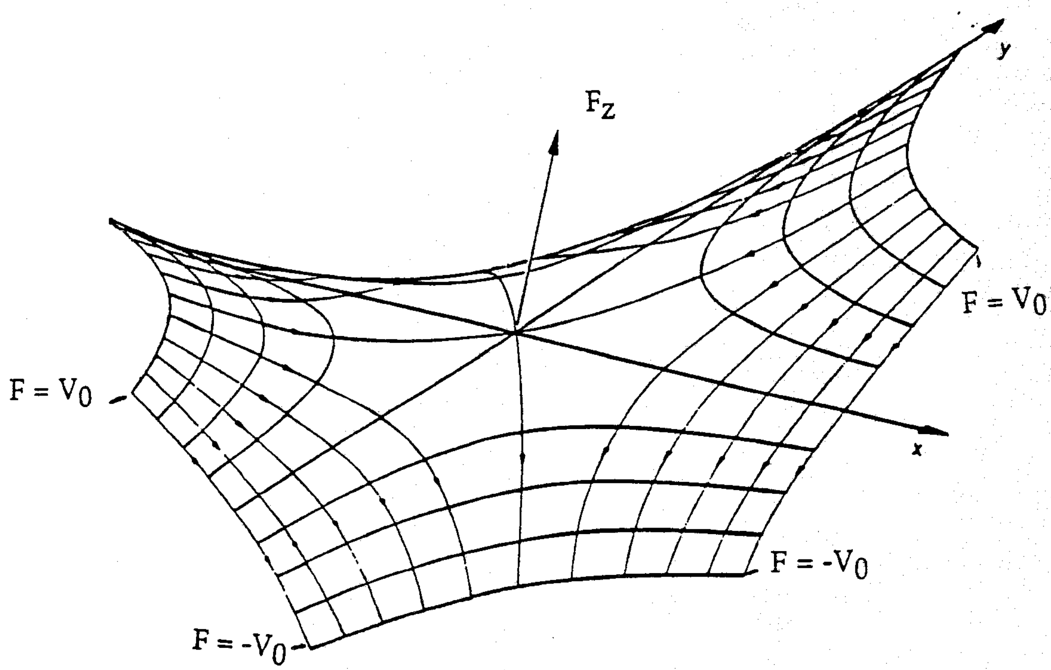
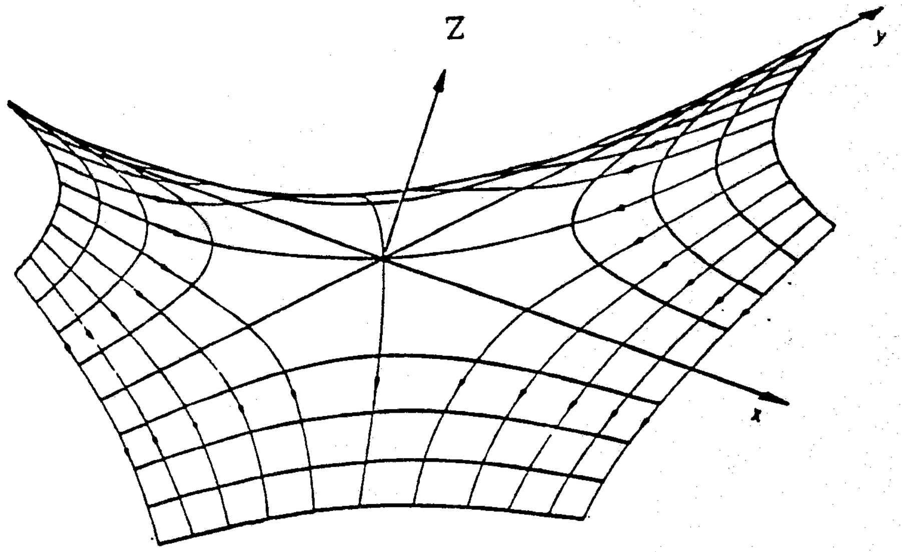
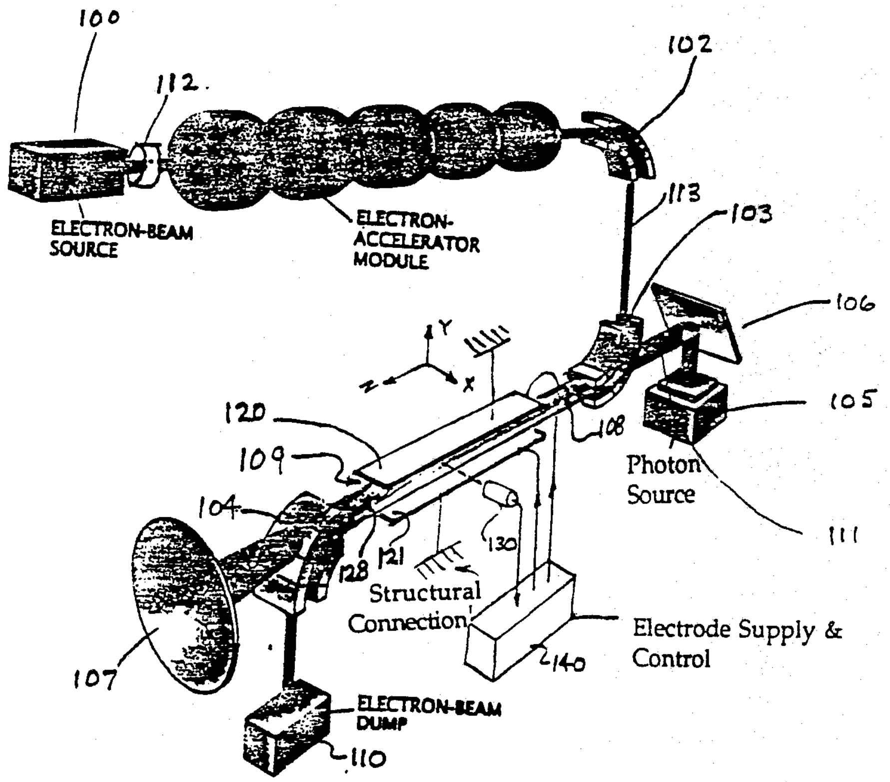


Figure 3.



4/8

Figure 4.


100
112.
102
ELECTRON-BEAM SOURCE
ELECTRON-ACCELERATOR MODULE
113
103
106
Y
Z
X
105
120
108
109
Photon Source
104
130
111
128
121
Structural Connection
Electrode Supply & Control
140
107
ELECTRON-BEAM DUMP
110

5/8

Figure 5.

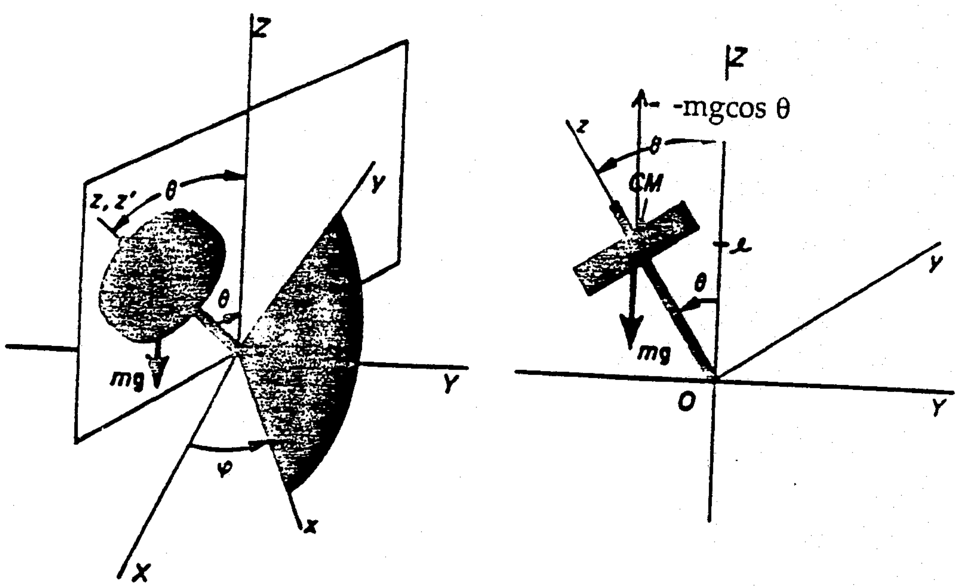
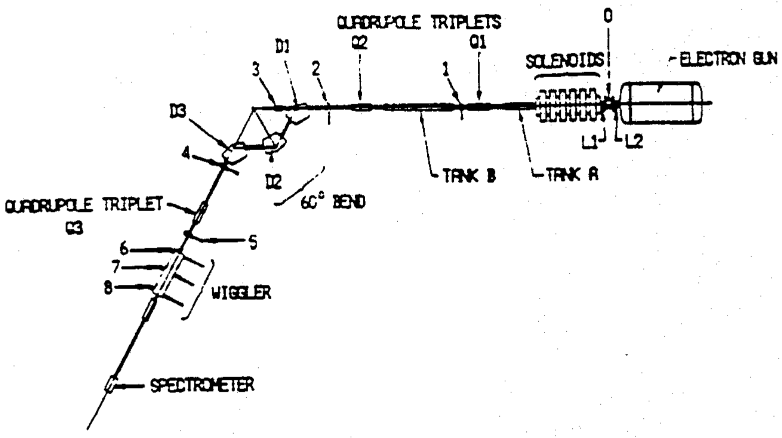


Figure 6.



QUADRUPOLE TRIPLETS
Q2
Q1
0
D1
3
2
1
SOLENOIDS
ELECTRON GUN
D3
L1
L2
4
D2
TANK B
TANK A
60° BEND
QUADRUPOLE TRIPLET
Q3
5
6
7
8
WIGGLER
SPECTROMETER

Figure 7. 7/8

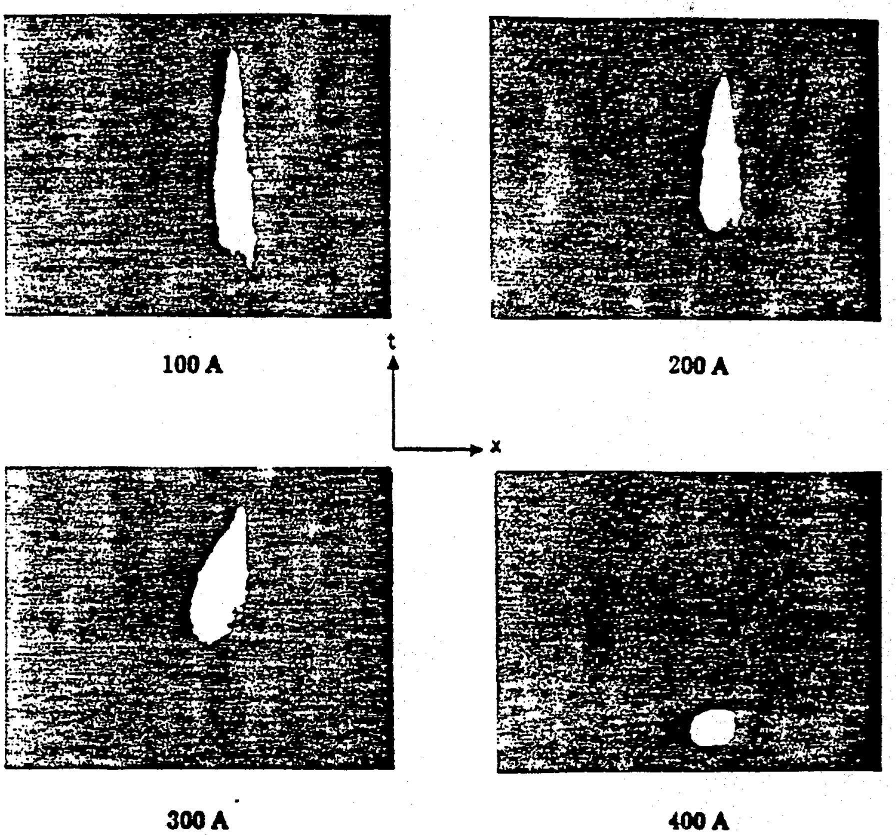
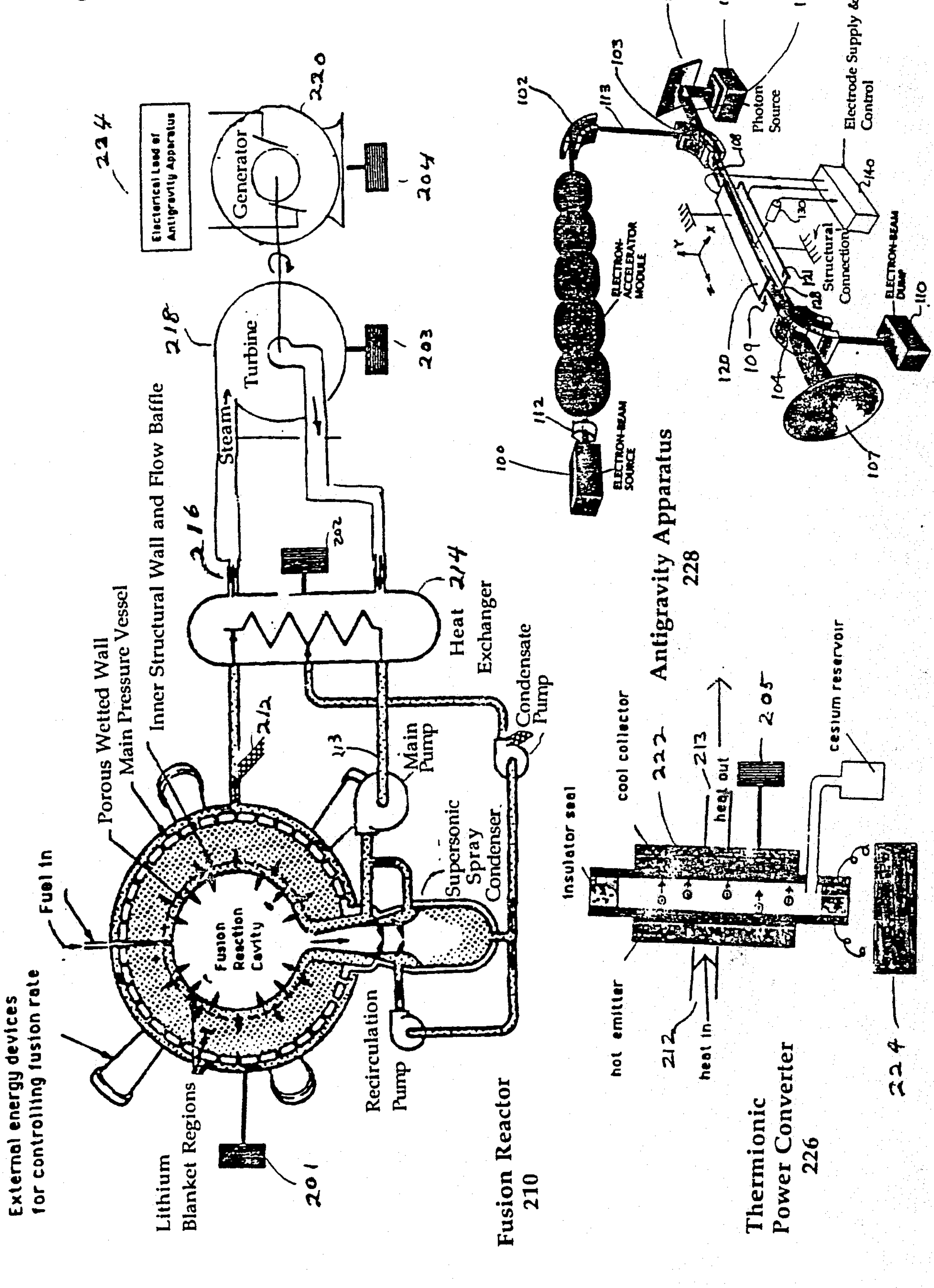


Figure 8.


External energy devices for controlling fusion rate
Lithium Blanket Regions
Fuel In
Porous Wetted Wall
Main Pressure Vessel
Inner Structural Wall and Flow Baffle
Fusion Reaction Cavity
201
Recirculation Pump
Supersonic Spray Condenser
Main Pump
Condensate Pump
212
213
Heat Exchanger
214
202
216
Steam
Turbine
218
203
Generator
220
204
224
Electrical Load of Antigravity Apparatus
Fusion Reactor
210
Thermionic Power Converter
226
hot emitter
heat in
insulator seal
cool collector
222
heat out
205
cesium reservoir
Antigravity Apparatus
228
ELECTRON-BEAM SOURCE
ELECTRON-ACCELERATOR MODULE
ELECTRON-BEAM DUMP
Photon Source
Electrode Supply & Control
Structural Connection
100
102
103
104
105
106
107
108
109
110
111
112
113
120
121
128
130
140

INTERNATIONAL SEARCH REPORT

International Application No PCT/US 90/03441

I. CLASSIFICATION OF SUBJECT MATTER (if several classification symbols apply, indicate all)[6]

According to International Patent Classification (IPC) or to both National Classification and IPC

Int.Cl. 5 G21K1/00

II. FIELDS SEARCHED

Minimum Documentation Searched[7]

Classification System	Classification Symbols
Int.Cl. 5	G21K ; H02N

Documentation Searched other than Minimum Documentation to the Extent that such Documents are Included in the Fields Searched[8]

III. DOCUMENTS CONSIDERED TO BE RELEVANT[9]

Category °	Citation of Document, [11] with indication, where appropriate, of the relevant passages [12]	Relevant to Claim No.[13]
A	NUCLEAR INSTRUMENTS & METHODS IN PHYSICS RESEARCH, SECTION A vol. A272, no. 1,2, November 1988, AMSTERDAM pages 247 - 256; CARLSTEN ET AL: "EMITTANCE STUDIES AT THE LOS ALAMOS NATIONAL LABORATORY FREE ELECTRON LASER" see page 252, right-hand column, paragraph 1 - page 253; figure 11 (cited in the application) ---	1, 10
A	SCIENTIFIC AMERICAN. vol. 258, no. 3, March 1988, NEW YORK US pages 32 - 40; T GOLDMAN ET AL: "GRAVITY AND ANTIMATTER" see page 39, last paragraph - page 40 ---	1, 10

-/--

° Special categories of cited documents : [10]

"A" document defining the general state of the art which is not considered to be of particular relevance

"E" earlier document but published on or after the international filing date

"L" document which may throw doubts on priority claim(s) or which is cited to establish the publication date of another citation or other special reason (as specified)

"O" document referring to an oral disclosure, use, exhibition or other means

"P" document published prior to the international filing date but later than the priority date claimed

"T" later document published after the international filing date or priority date and not in conflict with the application but cited to understand the principle or theory underlying the invention

"X" document of particular relevance; the claimed invention cannot be considered novel or cannot be considered to involve an inventive step

"Y" document of particular relevance; the claimed invention cannot be considered to involve an inventive step when the document is combined with one or more other such documents, such combination being obvious to a person skilled in the art.

"&" document member of the same patent family

IV. CERTIFICATION

Date of the Actual Completion of the International Search	Date of Mailing of this International Search Report
18 OCTOBER 1990	08 NOV 1990
International Searching Authority	**Signature of Authorized Officer**
EUROPEAN PATENT OFFICE	HULNE S.L.

Form PCT/ISA/210 (second sheet) (January 1985)

III. DOCUMENTS CONSIDERED TO BE RELEVANT (CONTINUED FROM THE SECOND SHEET)

Category °	Citation of Document, with indication, where appropriate, of the relevant passages	Relevant to Claim No.
A	REVIEW OF SCIENTIFIC INSTRUMENTS. vol. 48, no. 1, January 1977, NEW YORK US pages 1 - 11; WITTEBORN F C ET AL: "APPARATUS FOR MEASURING THE FORCE OF GRAVITY ON FREELY FALLING ELECTRONS" see page 1, paragraph 1 ---	1, 10
A	APPLIED PHYSICS. vol. A48, no. 1, January 1989, HEIDELBERG DE pages 87 - 91; MARSHALL D B: "FLUX PENETRATION IN HIGH-Tc SUPERCONDUCTORS:IMPLICATIONS FOR MAGNETIC SUSPENSION AND SHIELDING" see abstract; figure 1 ---	1, 10
A	DE,A,2456689 (BLUM J) 12 August 1976 see page 1, paragraph 3 - page 3, last paragraph ---	1, 10

ANNEX TO THE INTERNATIONAL SEARCH REPORT ON INTERNATIONAL PATENT APPLICATION NO.

US 9003441
SA 38307

This annex lists the patent family members relating to the patent documents cited in the above-mentioned international search report.
The members are as contained in the European Patent Office EDP file on 18/10/90
The European Patent Office is in no way liable for these particulars which are merely given for the purpose of information.

Patent document cited in search report	Publication date	Patent family member(s)	Publication date
DE-A-2456689	12-08-76	None	

EPO FORM P0479

For more details about this annex : see Official Journal of the European Patent Office, No. 12/82